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Treatise

Numbers-Based Musical Harmony

Simon Martin

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Treatise

Summary

In the West, the study of musical harmony through ratios of integers goes back as far as Pythagoras, some 2,500 years ago. This approach, which was superseded in the 19th century by the subdivision of the octave into twelve equal parts, is now enjoying renewed interest. The *Treatise* synthesizes some recent knowledge on this topic by means of a complete and coherent harmonic modulation method. The *Treatise* begins with simple basic notions to build toward the use of a vast repertoire of modulating harmonic sequences described strictly with integers available in the *Supplements*.

Supplements to the *Treatise* available at projectionsliberantes.ca.

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Table of Contents

Part Two

List of Tables

Glossary

Cent (abbrev. "c") — Unit of measurement which corresponds to the subdivision of each of the twelve equal divisions of the octave into one hundred equal parts. *E.g. 1 octave = 1,200 cents. 1 cent = 1/1,200 of an octave.*

Characteristic identity — Identity at the denominator of a tonality. The designation of a tonality takes the colour of its characteristic identity. *E.g. the characteristic identity of tonality 16/11* is 11.*

Denominative — Ratio between 1/1 and 2/1 used to designate an element by its function, notwithstanding any other consideration. *E.g. the ratios 5/1, 5/2, and 5/8 all share the denomination 5/4.*

Fundamental — The lowest common denominator of a set of harmonics. The 1-harmonic of that set. By extension, the fundamental designates the 1-identity of a chord, that is to say both the 1-harmonic and its octaves (2, 4, 8, etc.). *E.g. the fundamental of the 3-, 5-, and 7-harmonics is the lowest common denominator, i.e. the number 1. E.g. the fundamental is at the bass of the chord [4 : 5 : 6].*

Generating harmonic — Identity at the numerator of a tonality. *E.g. the numerator of the tonality 8/7* is 8, whose identity is 1. The generating harmonic of the tonality 8/7* is therefore 1.*

Harmonic — A pitch is said to be "harmonic" when it can be compared to another pitch by a ratio of whole numbers, or integers, of frequencies. A harmonic pitch (qualifier) is called a "harmonic" (noun). Harmonics that are integer multiples of a fundamental are the elements of that fundamental's tonality. As an element of a tonality, a harmonic occupies the order corresponding to its multiple. *E.g. the multiple 6 of the 1-fundamental generates the 6-order harmonic.*

Harmonic relationship — Ratio of integer frequencies.

Identity — Characteristic tonal function, identifiable on hearing it. The identities of a tonality correspond to the odd harmonics of the fundamental. In the *Treatise*, we assign a colour to each identity according to its greatest prime factor (beyond 3): 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 23, 25, etc. The identity of even harmonics corresponds to the identity of their greatest odd prime factor, whose colour they adopt since they constitute the various redoublings of octaves. *E.g. the 5-, 10-, and 20-harmonics are all 5-identities.*

Limit — Characterization of a system by the largest prime number involved. By extension, the limit sometimes relates specifically to the designated prime number. *E.g. a 5-limit system does not involve any prime numbers other than 2, 3 and 5. E.g. the 11-limit accidental symbol relates to the 11-identity.*

Modulation — A switch from one tonality to another.

Modulation factor — Interval between two tonalities designated by a denominative preceded by a multiplier sign (represented by an asterisk "*"). *E.g. the modulation factor between tonalities 4/3* and 8/5* is *6/5.*

Order of tonalities — Families of tonalities sharing the same generating harmonic. *E.g. the tonality families of order 7 are: 7/4*, 7/5*, 14/11*, etc.*

Ratio — Comparison of integer frequencies as a fractional number (x/y) or a proportion (x : y).

We use fractional numbers primarily to denote pitches. The x/y ratio thus implies a relation to a 1/1 generating pitch. However, fractional numbers can also be used to represent an interval, mainly when no specific harmonic is involved. *E.g. the doubled octave of the pitch 7/8 is 7/4. E.g. the pitch 1/1, multiplied by 5 and divided by 7, is equal to the pitch 5/7. E.g. to transpose a pitch by an ascending octave, we multiply it by the interval 2/1.*

We use proportions to denote intervals and chords. The numbers in proportions can be arranged in descending order when needed, but they are arranged in ascending order by convention, much like the way we spell out the notes of an interval (major third *C-E*) or chord (major triad *C-E-G*). *E.g. the interval between harmonics 5 and 4 is 4 : 5. E.g. the chord consisting of harmonics 2, 6, and 7 is 2 : 6 : 7.*

Subharmonic — Integer divider of a frequency. *E.g. the 5-subharmonic of 1 is 1/5.*

Tonal network — Set of tonalities for which 1/1 is the generating pitch.

Tonality — Feeling of coherence emerging from a group of pitches in harmonic relationships. A tonality is designated by the denominative of the fundamental, followed by a multiplier sign and adopting the colour of its identity at the denominator position. *E.g. the tonality whose fundamental is the 17-subharmonic of the 1-harmonic (ratio 1/17) is 32/17*.* A tonality whose identity at the denominator position is 1 or 3 is highlighted in grey, which makes it possible to differentiate between the denominative of the tonality and any other ratio written in black that does not have that function. *E.g. the ratio 4/3 has a general meaning, while the tonality 4/3* has a specific meaning.*

Tonality family — Set of tonalities which are multiples of 3 between them. The family is designated by the tonality whose factor 3 is absent. *E.g. the tonalities 16/9*, 4/3*, 1/1*, and 3/2* belong to the tonality family 1/1*.*

Acknowledgments

The *Treatise* would never have seen the light of day without the work done before me by three composers in particular.

Harry Partch (1901-1974) laid the modern foundations of numbers-based harmony, notably through his book *Genesis of a Music*[1].

Marc Sabat (1965-), in addition to publishing many theoretical writings, created, with Wolfgang von Schweinitz, the "Helmholtz-Ellis Just Intonation" (HEJI) notation system, the main aspects of which are described in chapter 2 of this book. Thomas Nicholson helped update the notation system in 2020, in addition to designing a web calculator to facilitate its use.

Robin Hayward (1969-) designed and developed the Hayward Tuning Vine software. This software, indispensable to my research since 2016, has proven to be a fantastic practical and educational tool. It is with this in mind that I adopted its colour code for identifying prime numbers. More information on how to use the software in conjunction with the *Treatise* can be found in the Appendix.

On a personal note, I would like to thank my wife who supports me in all my projects.

[1] The complete references of the works mentioned in the *Treatise* are presented in the References section at the end of the book.

To my people, the people of Quebec. To my country, Quebec.

For my part, I believe that the values of the future are not created. *They spring to light like plants, through an obscure and slow maturation process deep in the silence of the soil. On this, Quebec has something unique to say. This is what I call* independence*: a combination, in this case, of creativity and memory.*

Fernand Dumont, translated by François Couture

Introduction

Do, ré, mi, fa, sol, la, si, do.

Many people know the names in this sequence of notes without any knowledge of music. Most of them are probably unaware that this is a major scale.

1/1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/8, 2/1.

Many musicians do not know how to name this sequence of ratios. Most of them are probably unaware that this is a major scale.

We forget it, but beyond the notes are the numbers at the foundations of harmony. In the West, this knowledge would have been established some 2,500 years ago, when Pythagoras studied the proportions of a vibrating string, in particular by using the prime numbers 2 and 3. In his work entitled *Harmonics*, dating from the 2nd century, Claude Ptolemy integrated the number 5, which allowed him to divide a string in correspondence with the relationships described above.

2, 3, 5. These three numbers are enough to design most Western music scales. Then, in the 19th century, the division of the octave into twelve equal parts replaced proportions, with the consequence of confining the study (and teaching) of harmony to that of an enclosed twelve-note system. In the middle of the 20th century, the American composer Harry Partch reintroduced the number as a tool for investigating musical harmony. He called "*Just Intonation*" the system based on the auditory perception of simple ratios, such as those already presented, precisely because of our capacity to recognize the correctness of intonation of the musical pitches resulting from them. Despite the growing fame of his work, the path he proposed remains unknown beyond insider circles and difficult to access for those who do not master their jargon.

The *Treatise* synthesizes some recent knowledge on this topic by means of a complete and coherent harmonic modulation method. The book is divided into two parts. Part One, chapters 1 to 3, outlines most of the theoretical subject matter. Chapter 1, probably the most difficult chapter in the book, introduces all the fundamentals needed to approach numbers-based harmony. Next, Chapter 2 explains how to transcribe whole number ratios into musical notation. Finally, Chapter 3 explains how to string together numbers-based chords.

Part Two of the *Treatise*, chapters 4 through 7, contains lists of general tables, such as those describing all the notes and chords we use.

The *Treatise* is primarily intended for professionals working in the field of contemporary instrumental music. However, we have taken care to keep things accessible to anyone curious about this topic. To this end, each concept is explained when it first appears, without musical staves or complex mathematical formulas. The most specialized concepts, or those specific to the *Treatise*, are also included in a glossary. A vast repertoire of modulating harmonic sequences described strictly using whole numbers is available separately, as *Supplements* to the *Treatise*. These documents will thus allow readers to learn about the study of numbers-based harmony as well as to practise their expertise.

1

Part One

1. Basics

Chapter 1 introduces the basics of numbers-based harmony that will be useful to the beginner reader in this area, while allowing expert readers to familiarize themselves with the approach specific to the *Treatise*, in particular, with regard to the modulation factor discussed in section 1.8.

1.1 Basic concepts

1.1.1 Sound

Sound is the auditory sensation that results from pressure variations in the environment. Variations occurring at a frequency between about 20 and 20,000 times per second are captured by the human ear, converted into action potential (nerve impulses) and brought to consciousness.

A pressure differential that recurs at a periodic (regular) frequency is interpreted as a more or less low (slow frequency) or high (fast frequency) pitch. We say that pitches “go up” towards the high end and “go down” towards the low end in conjunction with the required throat movements to sing them. On the Western musical staff, the pitch is expressed symbolically by a more or less low or high note.

1.1.2 Harmony

In music, harmony refers to the ratios of integer numbers of frequencies, called “harmonic relationships.” Take the example of a first frequency “f” and a second frequency “2f” that is twice the first. The ratio of the second frequency to the first is 2f/f, or 2/1. 2/1 being a ratio of whole numbers of frequencies, we are in the presence of a harmonic relationship. A pitch is said to be “harmonic” when it can be compared to another pitch by a ratio of whole numbers, or integers, of frequencies. A harmonic pitch (qualifier) is called a “harmonic” (noun).

A ratio of integers of frequencies also represents the measure of the difference between two pitches, which is called an “interval.” The interval is melodic if the pitches are produced successively or harmonic if they are produced simultaneously. A harmonic interval can also be called a “dyad.” A group of three or more pitches sounding simultaneously is called a “chord.”

A three-tone chord is called a “triad”; a four-tone chord is called a “tetrad.” In reference to choral singing, the four pitches of a tetrad are commonly called, from the highest to the lowest, the “soprano,” “alto,” “tenor,” and “bass” voices (S, A, T, B). In this book, a generic triad omits the soprano voice.

1.1.3 Tonality

Tonality is the feeling of coherence emerging from a group of pitches in harmonic relationships. The pitches of a tonality are defined and organized as follows. The fundamental (qualitative) frequency is called the “fundamental” (substantive) of a tonality. The fundamental corresponds to the least common divisor of a set of harmonics. Conversely, harmonics that are integer multiples of a fundamental are that fundamental’s harmonics, which are also the elements of that fundamental’s tonality. The strength of the feeling of coherence between a harmonic and its fundamental depends on the simplicity of their frequency ratio: the simpler the ratio, the stronger the feeling of coherence. This is true for any pair of pitches.

1.1.4 Consonance

The 1 : 1 frequency ratio is a unison, or prime interval. Since the two frequencies are the same, a single pitch is perceived. If we double one of the two frequencies, we get the 1 : 2 ratio, which is the octave interval. Since the two frequencies are not the same, two pitches are distinguishable; however, the frequency coordination is so great that we perceive these two pitches to be the same note transposed in the register. This is the case, for example, when a choir of men and women sing a melody in octaves. The phrase “singing in unison” applies to this situation.

The 100 : 101 frequency ratio is perceived as an out-of-tune unison, i.e. a simple interval that is slightly inaccurate, rather than an exact complex interval. This tells us that the sense of tonal coherence or attraction exists not only between heard pitches, but also between implied pitches. Indeed, the more complex a frequency ratio is, the more we tend to interpret it as a simpler similar ratio[1]. Consequently, the simpler the frequency ratio (understood or implied), the wider and stronger its tonal field of attraction, so that any interval can be considered as a relative dissonance if it is in the tonal field of attraction of an interval of greater relative consonance[2].

[1] On this subject, see Hasegawa, 2006.

[2] For a deeper understanding of the notions of consonance and dissonance throughout history, see Tenney, 1988. Moreover, consonance depends on several factors other than the frequency ratio alone (in particular, the pitch register and the tone of the sounds involved), but this goes far beyond the scope of this book.

1.2 Identities

1.2.1 Harmonics

Let it be a fundamental of f frequency. Let us consider its integer multiples by limiting ourselves to 28[3]:

f, 2f, 3f, 4f, 5f, 6f, 7f, 8f, 9f… 28f.

These integer multiples are harmonics of orders 1 to 28 and constitute elements of the 1-tonality:

1, 2, 3, 4, 5, 6, 7, 8, 9… 28.

Among these multiples, even harmonics represent octaves: 2 is the octave of 1 (2/1); 4 is the octave of 2 (4/2 = 2/1); 6 is the octave of 3 (6/3 = 2/1); 8 is the octave of 4, which is the octave of 2, which in turn is the octave of 1 (8/4 = 4/2 = 2/1); etc.

Therefore, even numbers do not represent any new note in the tonality; in other words, no new tonal functions. So let us consider only the odd harmonics of this series:

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27.

The odd harmonics of a fundamental constitute the "identities[4]" of its tonality, that is to say its characteristic tonal functions, identifiable on hearing. Among these odd harmonics are prime and non-prime harmonics. Prime harmonics have for factors only 1 and themselves. Prime harmonics are therefore tonal identities exclusive to the 1 fundamental. For example: the only factors of 3 are 1 and 3; 3 is a prime harmonic; the 3-harmonic is the tonal identity 3, exclusive to the 1-fundamental.

The prime harmonics of the series are[5]:

1, 3, 5, 7, 11, 13, 17, 19, 23.

[3] We agree with Hayward and Sabat, 2006, taken up by Nicholson, 2021, about the perceptibility of harmonics, which would be much more imprecise starting with the prime number 29.

[4] This term and its meaning come directly from Partch, 1974, p. 71.

[5] "1" is not a prime harmonic, but it is used throughout this work to represent the prime harmonic "2," which could cause confusion as an even harmonic, octave of "1."

To better identify them in the future, let us assign a colour[6] to each of these exclusive identities:

1, 3, 5, 7, 11, 13, 17, 19, 23.

The largest prime number involved in a system is considered its "limit[7]." For example, the harmonics that we are observing constitute a "23-limit" system.

Odd non-prime harmonics have several prime factors. For example, the prime factors of 15 are 1, 3, and 5. So, harmonic 15 is an identity for several potential tonalities:

— the 15-identity of 1-tonality (15/1 = 15);
— the 5-identity of 3-tonality (15/3 = 5/1 = 5);
— the 3-identity of 5-tonality (15/5 = 3/1 = 3).

The odd non-prime harmonics of the series are:

9, 15, 21, 25, 27.

Let us assign a colour to these non-exclusive identities based on their highest prime factor:

— 9 and 27 have the number 3 as their greatest prime factor;
— 15 and 25 have the number 5 as their greatest prime factor;
— 21 has the number 7 as its greatest prime factor.

Similarly, the identity of even harmonics corresponds to the identity of their greatest prime factor, of which they take the colour.

— 5, 10, 20...
— 7, 14, 28...
— 11, 22...
— etc.

In the light of these principles, we can now organize the harmonics of orders 1 to 28 in a table according to their identity (horizontally) and their octave (vertically).

[6] With the exception of multiples 2 and 3 which we will keep in black, this colour code is the same one used in the Hayward Tuning Vine software designed by Robin Hayward (see Appendix). About this software, see Hayward, 2015.

[7] The common use of this term, used in this sense, comes directly from Partch, 1974, p. 109.

5-octave	16	17	18	19	20	21	22	23	24	25	26	27	28	
4-octave	8		9		10		11		12		13		14	15
3-octave	4				5				6				7	
2-octave	2								3					
1-octave	1													

Table 1.1: Harmonics 1 to 28 classified according to their identity and their octave

We can also now transcribe a chord by its harmonics grouped in proportions and preceded by the fundamental of which they are multiple. For example, here is how to transcribe the chord consisting of the 1-3-5 harmonics in 1-tonality:

1*[1:3:5]

For another example, consider a chord composed of the 5-15-25 harmonics:

[5:15:25]

Exceptionally, these harmonics have a common divider greater than 1, i.e. 5:

$$\frac{[5 : 15 : 25]}{5} = \frac{[1 : 3 : 5]}{1}$$

The chord therefore belongs to the tonality of 5:

5*[1:3:5]

Analyzing harmonics according to their tonality makes it easier to recognize chords, the recurring pattern of which becomes that of the simplest expression of their periodicity. Indeed, heard out of context, the [5:15:25] harmonics are perceived according to the proportion of their frequencies: [1:3:5].

1.2.2 Subharmonics

In addition to representing the measure of an interval, the x/y ratio accounts for the duality of any pitch in a tonal context; that is, the potential for any pitch to be the fundamental or harmonic of a tonality.

We have seen that the identities of a tonality correspond to odd harmonics, which are integer multiples of the fundamental. For example, to find the 5-identity of the 1-fundamental, we must multiply 1 by 5:

$1 \times 5 = 5$

Conversely, the identities of a harmonic pitch correspond to its odd subharmonics, which are integer dividers. For example, to find the tonality in which the harmonic pitch 1 is a 5-identity, divide 1 by 5:

$1 \div 5 = \frac{1}{5}$

Therefore, 1/5 constitutes the fundamental of the tonality for which the harmonic pitch 1 is a 5-identity. As proof, if we multiply the 1/5-fundamental by the sought identity, 5, we find the harmonic pitch 1:

$$\frac{1}{5} \times 5 = \frac{1}{5} \times \frac{5}{1} = \frac{1 \times 5}{5 \times 1} = \frac{5}{5} = \frac{1}{1} = 1$$

The subharmonics thus generate as many fundamentals as there are conferred identities at the divided harmonic pitch. For example, the tonalities in which the harmonic pitch 1 is an identity are as follows:

$$\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11} \ldots$$

1.3 Graphical representation

Representing harmonic relationships graphically[8] helps us visualize them.

1.3.1 Horizontal axis

For example, let us first have horizontally a few multiples of 3 of the 1/1 pitch, called the "generating pitch," to obtain its 3-, 9-, and 27-harmonics:

harmonics of 1/1 →

$\frac{1}{1}$	$\frac{3}{1}$	$\frac{9}{1}$	$\frac{27}{1}$

Table 1.2: Harmonics multiples of 3 of the 1/1-fundamental

This sequence of multiples also extends to the left to make 1/1 the 3rd, 9th, and 27th harmonics of a tonality:

← subharmonics de 1/1 **harmonics of 1/1 →**

$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{1}$	$\frac{3}{1}$	$\frac{9}{1}$	$\frac{27}{1}$

Table 1.3: Harmonics multiples of 3, starting from 1/27

Each pitch in the sequence can also be considered as a fundamental (1-harmonic), with its harmonics multiples of 3 to its right. Here are a few examples:

identity	1	3	9	27	81	243	729
ratio	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{1}$	$\frac{3}{1}$	$\frac{9}{1}$	$\frac{27}{1}$

Table 1.4: Identities multiples of 3 of the 1/27-fundamental

identity	1	3	9	27	81	243
ratio	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{1}$	$\frac{3}{1}$	$\frac{9}{1}$	$\frac{27}{1}$

Table 1.5: Identities multiples of 3 of the 1/9-fundamental

[8] See Tenney, 2008. See also the page entitled "Tonnetz" in Wikipedia:https://en.wikipedia.org/wiki/Tonnetz (accessed March 12, 2021).

identity	1	3	9	27	81
ratio	1/3	1/1	3/1	9/1	27/1

Table 1.6: Identities multiples of 3 of the 1/3-fundamental

Finally, these examples also allow us to see that the denominator of the fundamental indicates the identity of the 1/1 generating pitch in the tonality.

1.3.2 Vertical axis

Then, let us lay out vertically the multiples of 5 for each of these pitches:

			25/1			
5/27	5/9	5/3	5/1	15/1	45/1	
1/27	1/9	1/3	1/1	3/1	9/1	27/1
	1/45	1/15	1/5	3/5	9/5	27/5

Table 1.7: Harmonics and subharmonics multiples of 3 and 5

To reach 15/1 starting from the 1/1 generating pitch, one can visualize two paths: multiply 1/1 by 5 to find 5/1 upwards, then multiply 5/1 by 3 to find 15/1 to the right, or; multiply 1/1 by 3 to find 3/1 to the right, then multiply 3/1 by 5 to find 15/1 upwards.

The graphical representation thus makes it possible to easily deduce links from identity to tonality. To continue with the previous example, 15/1 is

— the 15-identity of 1/1 tonality (1*15 = 15);
— the 5-identity of 3/1 tonality (3*5 = 15);
— the 3-identity of 5/1 tonality (5*3 = 15).

We can also visualize that the relationship between 1/1 and 15/1 is the same as the relationship between 1/15 and 1/1, or any other pair of ratios arranged in the same configuration.

1.3.3 Diagonal axis

Finally, let us add the multiples of 7 diagonally. For example, we can visualize that the relationship of 7/3 to 7/1 is the same as the relationship of 1/3 to 1/1, or of 1/21 to 1/7, etc. Also, we can easily calculate that 7/1 is the 49-harmonic (7*7) of 1/7, since to move from 1/7 to 7/1, one has to multiply 1/7 by 7 twice to go up twice diagonally.

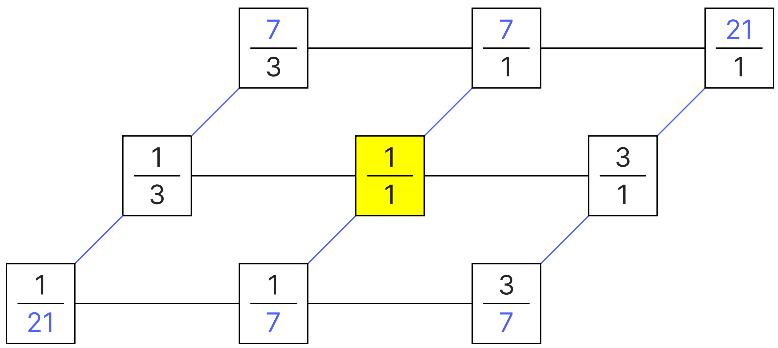


Table 1.8: Harmonics and subharmonics multiples of 3 and 7

In short, the three multiples (3, 5 and 7) can be represented simultaneously in the same graph (Table 1.9). Again, the graphical representation makes it easier to deduce harmonic relationships. For example, to find 5/7 starting from the 1/1 generating pitch, one visualizes that it is necessary to multiply 1/1 by 5 to find 5/1 upwards, then divide 5/1 by 7 to find 5/7 diagonally downwards.

In addition, the numerator of any harmonic pitch tells us its identity in a tonality, while its denominator tells us the identity of 1/1 in that same tonality. To continue with the example of the 5/7 harmonic pitch, the numerator indicates its identity (5, in the tonality of 1/7) and its denominator indicates the identity of 1/1 in that same tonality (7).

Any multiples other than 3, 5, and 7 can be arranged horizontally, vertically, and diagonally, and extend in either direction as needed. Different graphs can thus be used to represent the harmonic network of different sections of a work. Other indications can be added to the graph, such as the names of the corresponding notes.

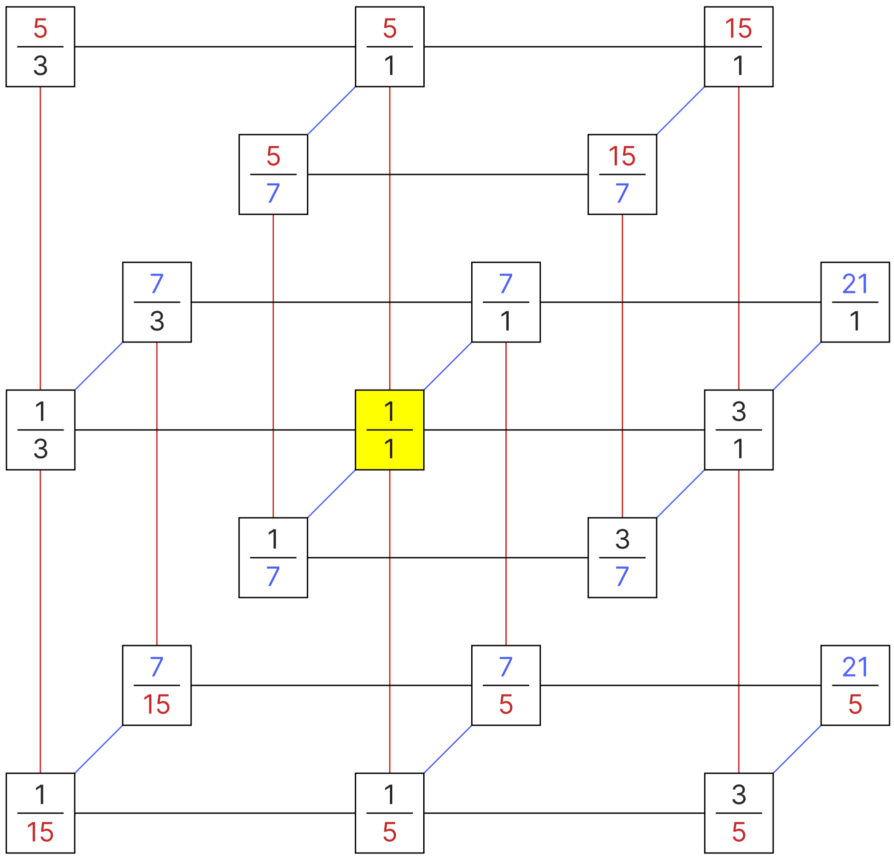


Table 1.9: Harmonics and subharmonics multiples of 3, 5 and 7

1.4 Denominative

The denominative is a ratio between 1/1 and 2/1 used to designate an element by its function, notwithstanding any other consideration. For example, in conventional harmony, when we list the notes *do*, *ré*, *mi*, *fa*, *sol*, *la*, *si*, *do*, it is implied that these notes are within the same octave and that they are not randomly distributed in the range. In this context, the repetition of *do*, just like that of the ratio 2/1, indicates the periodicity of the sequence, identical for each octave. In numbers-based harmony, the denominative of the ratios makes it possible to designate certain elements, including pitches, in the same way.

1.4.1 Calculation

For example, to find the denominative of the 14/8 ratio, which is 7/4, one must decompose its terms into prime numbers to find its identities (by dividing even numbers by 2 until you find odd numbers),

$$\frac{14 \div 2}{8 \div 2^3} = \frac{7}{1}$$

then, multiply the smallest term by 2 to transpose its octave until the ratio lands between 1/1 and 2/1.

$$\frac{7}{1 \times 2^2} = \frac{7}{4}$$

For another example, to find the denominative of the 16/46 ratio, which is 32/23, one must decompose its terms into prime numbers to find its identities,

$$\frac{16 \div 2^4}{46 \div 2} = \frac{1}{23}$$

then, multiply the smallest term by 2 to transpose its octave until the ratio lands between 1/1 and 2/1.

$$\frac{1 \times 2^5}{23} = \frac{32}{23}$$

Calculating the denominative of a ratio requires simplifying it by dividing each of its terms by the greatest common divisor when such a divisor exists. For example, for the ratio 10/15, this divisor is 5. Simplifying leads to the denominative 4/3:

$$\frac{10 \div 5}{15 \div 5} = \frac{2}{3}$$

$$\frac{2 \times 2}{3} = \frac{4}{3}$$

1.4.2 Pitches

Here is how to designate pitches by their denominatives.

For example, the table below presents: harmonic identities in ascending order; the relationship of these identities to the 1-fundamental; the denominative of the generated pitches. The different pitches of a tonality share the same identity at the denominator, which expresses their relationship to the same fundamental.

identity	1	3	5	7	9	11	13	15	17	19	21	23	25	27
$\frac{\text{identity}}{\text{1-fundamental}}$	$\frac{1}{1}$	$\frac{3}{1}$	$\frac{5}{1}$	$\frac{7}{1}$	$\frac{9}{1}$	$\frac{11}{1}$	$\frac{13}{1}$	$\frac{15}{1}$	$\frac{17}{1}$	$\frac{19}{1}$	$\frac{21}{1}$	$\frac{23}{1}$	$\frac{25}{1}$	$\frac{27}{1}$
denominative	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{7}{4}$	$\frac{9}{8}$	$\frac{11}{8}$	$\frac{13}{8}$	$\frac{15}{8}$	$\frac{17}{16}$	$\frac{19}{16}$	$\frac{21}{16}$	$\frac{23}{16}$	$\frac{25}{16}$	$\frac{27}{16}$

Table 1.10: Denominatives of pitches

To organize the denominative ratios of the previous table in ascending order and thus create a musical scale, all we need to do is to associate them with their harmonic order in the same octave; in this case, between harmonics 16 and 32 (since 32/16 = 2/1).

order	16	17	18	19	20	21	22	23	24	25	26	27	28	30	32
denominative	$\frac{1}{1}$	$\frac{17}{16}$	$\frac{9}{8}$	$\frac{19}{16}$	$\frac{5}{4}$	$\frac{21}{16}$	$\frac{11}{8}$	$\frac{23}{16}$	$\frac{3}{2}$	$\frac{25}{16}$	$\frac{13}{8}$	$\frac{27}{16}$	$\frac{7}{4}$	$\frac{15}{8}$	$\frac{2}{1}$

Table 1.11: Scaling of denominatives of pitches

1.4.3 Tonalities

Here is how to designate tonalities by their denominatives.

For example, the table below presents: subharmonic identities in ascending order; the relationship of these identities to the 1-harmonic; the denominatives of the generated tonalities. The different tonalities share the same identity at the numerator, which expresses their relationship to the same generating harmonic.

identity	1	3	5	7	9	11	13	15	17	19	21	23	25	27
1-harmonic / identity	1/1	1/3	1/5	1/7	1/9	1/11	1/13	1/15	1/17	1/19	1/21	1/23	1/25	1/27
denominative	1/1	4/3	8/5	8/7	16/9	16/11	16/13	16/15	32/17	32/19	32/21	32/23	32/25	32/27

Table 1.12: Denominatives of tonalities generated by the 1-harmonic

The denominative of a tonality is followed by a multiplier sign and adopts the colour of its denominator, which indicates the function of the generating harmonic in this tonality. The function of the generating harmonic of a tonality is characteristic of that tonality. For example:

— 1-harmonic is an 11-identity in tonality 1/11, whose denominative is 16/11*, since the characteristic identity of tonality 16/11* is 11;
— 5-harmonic is a 7-identity in tonality 5/7, whose denominative is 10/7*, since the characteristic identity of tonality 10/7* is 7;
— etc.

A tonality whose identity at the denominator position is 1 or 3 is highlighted in grey, which makes it possible to differentiate between the designation of the tonality and the tonality of any other ratio that does not have that function. E.g. the ratio 4/3 has a general meaning, while the tonality 4/3* has a specific meaning.

The denominative of the tonalities is obtained in the same way, regardless of the generating harmonic of the tonality. For example, the table below presents: subharmonic identities in ascending order; the relationship of these identities to 5-harmonic; the denominative of the generated tonalities. Tonalities 5/5* and 5/15*, respectively equivalent to 1/1* and 4/3*, have been removed so as to have only 5-harmonics at the numerator position.

identity	1	3	5	7	9	11	13	15	17	19	21	23	25	27
5-harmonic / identity	5/1	5/3	5/5	5/7	5/9	5/11	5/13	5/15	5/17	5/19	5/21	5/23	5/25	5/27
denominative	5/4	5/3	-	10/7	10/9	20/11	20/13	-	20/17	20/19	40/21	40/23	40/25	40/27

Table 1.13: Denominatives of tonalities generated by the 5-harmonic

To represent the relationships between a few tonalities, let us take again Table 1.9 and translate each pitch into a tonality denominative.

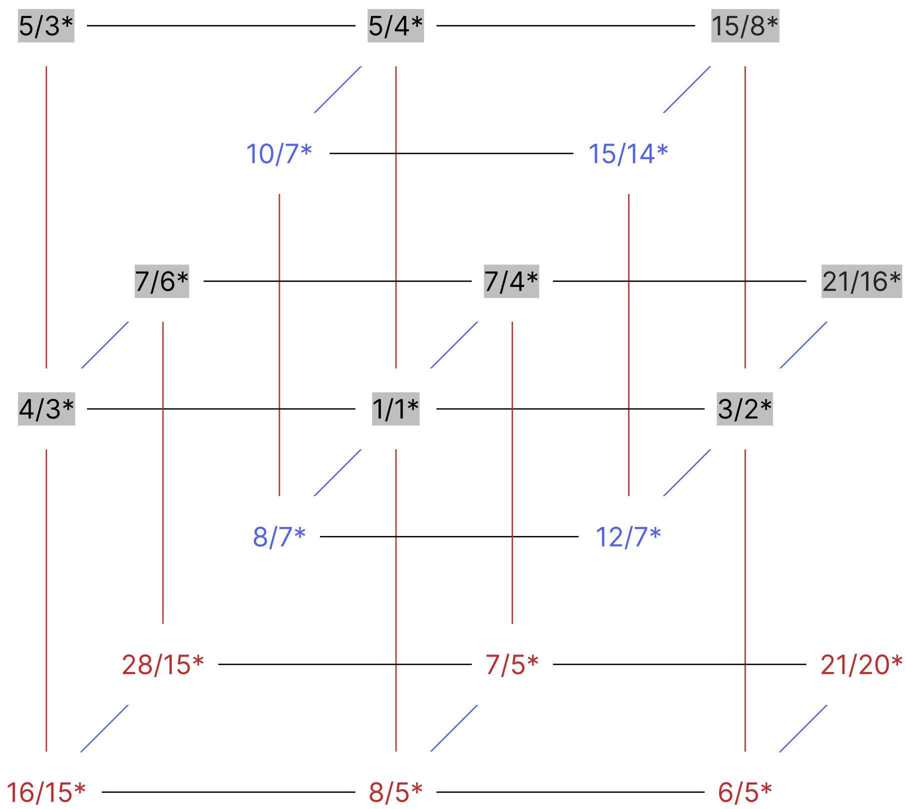


Table 1.14: Graphical representation of tonality denominatives

Finally, to find the denominative of an identity according to its tonality, taking for example 10/7*, one must consider the identities of the tonality,

10/7*[1 : 3 : 5 : 7 : 9 : 11...]

translate identities into fractional numbers,

$$10/7^*[\frac{1}{1} : \frac{3}{1} : \frac{5}{1} : \frac{7}{1} : \frac{9}{1} : \frac{11}{1} ...]$$

multiply the results by the denominative of the tonality

$$\frac{10 \times 1}{7 \times 1} : \frac{10 \times 3}{7 \times 1} : \frac{10 \times 5}{7 \times 1} : \frac{10 \times 7}{7 \times 1} : \frac{10 \times 9}{7 \times 1} : \frac{10 \times 11}{7 \times 1} \ ...$$

$$\frac{10}{7} : \frac{30}{7} : \frac{50}{7} : \frac{70}{7} : \frac{90}{7} : \frac{110}{7} ...$$

and translate the results into denominatives.

$\frac{10}{7} : \frac{15}{14} : \frac{25}{14} : \frac{5}{4} : \frac{45}{28} : \frac{55}{28}$ ...

1.5 Cents

The twelve notes of the conventional Western music system correspond to the division of the octave into twelve equal parts. Apart from the octave, the intervals of this system cannot be described by ratios of whole numbers, but only by irrational numbers. Each of these twelve equal parts of the octave is in turn subdivided into 100 units called "cents[9]," abbreviated "c." To switch from one octave to another, in this case, one must add a constant value (1,200) rather than multiply by a constant value (2/1).

The table below shows: the English names (letters) of the twelve notes of the conventional Western music system; the mathematical values of these notes in irrational numbers, and; their values in cents. The white and grey cells depict the corresponding piano key (white or black) of each note.

note	*C*	*C♯* *D♭*	*D*	*D♯* *E♭*	*E*	*F*	*F♯* *G♭*	*G*	*G♯* *A♭*	*A*	*A♯* *B♭*	*B*	*C*
irrational number	1	$2^{1/12}$	$2^{1/6}$	$2^{1/4}$	$2^{1/3}$	$2^{5/12}$	$\sqrt{2}$	$2^{7/12}$	$2^{2/3}$	$2^{3/4}$	$2^{5/6}$	$2^{11/12}$	2
cents	0	100	200	300	400	500	600	700	800	900	1000	1100	1200

Table 1.15: Values resulting from dividing the octave into twelve equal parts

Using a mathematical formula, it is possible to translate a ratio of whole numbers into cents, which makes it easier to conceptualization its size[10]. For example, here is how the twelve notes of the conventional system compare to some of the ratios we have just scaled[11].

[9] This unit was designed by Alexander J. Ellis. See Helmholtz, 1895.

[10] A ratio R corresponds to a number n of cents following that $1200 \times \log_2 (R) = n_{cents}$. See Nicholson and Sabat, 2018.

[11] For a list of many ratios organized in cents, see Gann, 1998.

Note the significant deviation (in cents) that exists between these intervals, obtained by proportions of whole numbers, and those obtained by the division of an octave into twelve equal parts presented in Table 1.15.

note	C	C♯ D♭	D	D♯ E♭	E	F	F♯ G♭	G	G♯ A♭	A	A♯ B♭	B	C
ratio	1/1	17/16	9/8	19/16	5/4	21/16	23/16	3/2	8/5	27/16	7/4	15/8	2/1
cents	0	105	204	298	386	471	628	702	814	906	969	1088	1200

Table 1.16: Comparison of the twelve conventional notes with a few ratios

A ratio may be accompanied by its value in cents put in parentheses. For example:

5/4 (386 c) or 4 : 5 (386 c)

When denoting the size of an ascending (positive value) or descending (negative value) interval, the value in cents is preceded by the corresponding sign. For example:

4 : 5 (+386 c) or 5 : 4 (-386 c).

In a score, we suggest indicating to the musician the cents value of melodic intervals less than about 32 : 33 (53 c), as these are performed more as an intonation adjustment, which is sometimes difficult to quantify, rather than a change of fingering. Any other melodic interval that may seem counter-intuitive, in particular due to the complexity of the accidental symbols involved (see chapter 2), can be specified in the same way.

1.6 Genesis of the tonal network

A tonal network is a set of tonalities for which 1/1 is the generating pitch. Once generated, the contents of the tonal network can be used freely, in whole or in part, just like the twelve notes of the conventional Western system. In numbers-based harmony, a tonal network theoretically extends to infinity. Therefore, arbitrary limits must be set for it depending on the identities we want to use and the musical instruments that will have to reproduce them. So let us generate a 23-limit tonal system, considering the natural resonance of a typical string instrument.

1.6.1 1-order tonalities

Firstly, the vibration of an open string (without fingering) produces the fundamental frequency of that string, i.e. 1. As we have done previously (see sections 1.2.2 and 1.4.3), let us give harmonic pitch 1 the identities 1, 5, 7, 11, 13, 17, 19 ou 23 by dividing it by the corresponding subharmonic.

$$\frac{1}{1}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \frac{1}{17}, \frac{1}{19}, \frac{1}{23}$$

These tonalities generated by the 1-harmonic are of 1-order.

1.6.2 5- and 7-order tonalities

Second, in addition to its fundamental frequency, a string can produce a number of its harmonics if it is lightly touched by a finger (rather than being pressed against the fingerboard)[12]. The harmonic obtained is proportional to the position of the finger in relation to the length of the string:

— 2-harmonic is obtained at half of the string's length;
— 3-harmonic at a third;
— 4-harmonic at a quarter;
— 5-harmonic at a fifth;
— 6-harmonic at a sixth;
— 7-harmonic at a seventh.

Let us now give harmonics 5 and 7 the identities 1, 5, 7, 11, 13, 17, 19 or 23 by dividing them by the corresponding subharmonics, so as to generate the tonalities of orders 5 and 7. In a table, let us gather the results by arranging the harmonics horizontally and the subharmonics vertically. The 5/5 and 7/7 ratios are not necessary since they are equivalent to 1/1.

[12] The same goes for the air column of an instrument of the brass family. For example, the trumpet resonates at its fundamental frequency when played without depressing the valves. Tightening the lips and blowing into the mouthpiece causes the air column to vibrate in harmonic mode (2, 3, 4, 5...). Pressing the valves transposes the fundamental of the air column and gives access to new series of harmonics. The combination of fingerings and the control of harmonics with the lips thus makes it possible to play musical scales.

$\frac{1}{}$	$\frac{5}{}$	$\frac{7}{}$	
$\frac{1}{1}$	$\frac{5}{1}$	$\frac{7}{1}$	$\frac{}{1}$
$\frac{1}{5}$	$\frac{5}{5}$	$\frac{7}{5}$	$\frac{}{5}$
$\frac{1}{7}$	$\frac{5}{7}$	$\frac{7}{7}$	$\frac{}{7}$
$\frac{1}{11}$	$\frac{5}{11}$	$\frac{7}{11}$	$\frac{}{11}$
$\frac{1}{13}$	$\frac{5}{13}$	$\frac{7}{13}$	$\frac{}{13}$
$\frac{1}{17}$	$\frac{5}{17}$	$\frac{7}{17}$	$\frac{}{17}$
$\frac{1}{19}$	$\frac{5}{19}$	$\frac{7}{19}$	$\frac{}{19}$
$\frac{1}{23}$	$\frac{5}{23}$	$\frac{7}{23}$	$\frac{}{23}$

Table 1.17: Factors of tonalities of orders 1, 5, and 7

The same table is repeated below with tonalities designated by their denominatives.

1/	5/	7/	
1/1*	5/4*	7/4*	/1
8/5*		7/5*	/5
8/7*	10/7*		/7
16/11*	20/11*	14/11*	/11
16/13*	20/13*	14/13*	/13
32/17*	20/17*	28/17*	/17
32/19*	20/19*	28/19*	/19
32/23*	40/23*	28/23*	/23

Table 1.18: Denominatives for tonalities of orders 1, 5, and 7

1.6.3 Tonality families

Third, in addition to the string of value 1 (or 1/1), let us add three more strings, each being the 3-identity of its neighbouring lower string.

string	factors	denominative
I	1/1	1/1*
II	1/3	4/3*
III	$1/3^2$	16/9*
IV	$1/3^3$	32/27*

Table 1.19: Tonalities of open strings I to IV

The 3-harmonic of the I string generates the 3/2* tonality, while the IV string, which can be considered a 3-harmonic of a tonality, generates the 128/81* tonality.

string	factors	denominative
	3/1	3/2*
I	1/1	1/1*
II	1/3	4/3*
III	$1/3^2$	16/9*
IV	$1/3^3$	32/27*
	$1/3^4$	128/81*

Table 1.20: Tonalities multiple of 3 generated by open strings I to IV

The 5- and 7-harmonics of each of the four strings do not themselves produce harmonics, but can be considered as the 3-harmonics of a tonality (respectively, 160/81* and 112/81*).

string	factors	denominative
I	5/1	5/4*
II	5/3	5/3*
III	$5/3^2$	10/9*
IV	$5/3^3$	40/27*
	$5/3^4$	160/81*

string	factors	denominative
I	7/1	7/4*
II	7/3	7/6*
III	$7/3^2$	14/9*
IV	$7/3^3$	28/27*
	$7/3^4$	112/81*

Table 1.21: Tonalities multiples of 3 generated by 5- and 7-harmonics of strings I to IV

The tonalities that are multiples of 3 between them form a family of tonalities. A tonality family is designated by the tonality whose factor 3 is absent, since it is the one that is closest to the 1/1 generating pitch. For example, the tonalities of the 5/4* family are: 5/4*, 5/3*, 10/9*, etc.

1.6.4 Tonal network

To group all the tonalities of the network in a single table, all we need to do is to add the tonalities of the same family from bottom to top within the same cell, while leaving tonality orders in columns. The importance of the table on the following page will be revealed later, particularly when explaining the modulation factor (see section 1.8).

	1/	5/	7/	
	3/2* —— 1/1* —— 4/3* 16/9* 32/27* 128/81*	- —— 5/4* —— 5/3* 10/9* 40/27* 160/81*	- —— 7/4* —— 7/6* 14/9* 28/27* 112/81*	/1
	6/5* —— 8/5* —— 16/15* 64/45* 256/135*		- —— 7/5* —— 28/15* 56/45* 224/135*	/5
	12/7* —— 8/7* —— 32/21* 64/63* 256/189*	- —— 10/7* —— 40/21* 80/63* 320/189*		/7
	12/11* —— 16/11* —— 64/33* 128/99* 512/297*	- —— 20/11* —— 40/33* 160/99* 320/297*	- —— 14/11* —— 56/33* 112/99* 448/297*	/11
	24/13* —— 16/13* —— 64/39* 128/117* 512/351*	- —— 20/13* —— 40/39* 160/117* 640/351*	- —— 14/13* —— 56/39* 224/117* 448/351*	/13
	24/17* —— 32/17* —— 64/51* 256/153* 512/459*	- —— 20/17* —— 80/51* 160/153* 640/459*	- —— 28/17* —— 56/51* 224/153* 896/459*	/17
	24/19* —— 32/19* —— 64/57* 256/171* 1024/513*	- —— 20/19* —— 80/57* 320/171* 640/513*	- —— 28/19* —— 112/57* 224/171* 896/513*	/19
	24/23* —— 32/23* —— 128/69* 256/207* 1024/621*	- —— 40/23* —— 80/69* 320/207* 640/621*	- —— 28/23* —— 112/69* 224/207* 896/621*	/23

Table 1.22: Summary table of the tonal network

The next two tables present all the numbers, organized by multiples, that need to be known in order to quickly locate a tonality in the tonal network.

Below, the prime numbers multiplied by 3 (tonality denominators):

/1	/1	/5	/7	/11	/13	/17	/19	/23
/3	/3	/15	/21	/33	/39	/51	/57	/69
$/3^2$	/9	/45	/63	/99	/117	/153	/171	/207
$/3^3$	/27	/135	/189	/297	/351	/459	/513	/621
$/3^4$	/81							
$/3^5$	/243							

Table 1.23: Prime numbers multiplied by 3

Below, the odd numbers multiplied by 2 (tonality numerators):

1/	1/	3/	5/	7/	9/	15/
2/	2/	6/	10/	14/	18/	30/
$2^2/$	4/	12/	20/	28/	36/	
$2^3/$	8/	24/	40/	56/		
$2^4/$	16/		80/	112/		
$2^5/$	32/		160/	224/		
$2^6/$	64/		320/	448/		
$2^7/$	128/		640/	896/		
$2^8/$	256/					
$2^9/$	512/					
$2^{10}/$	1024/					

Table 1.24: Odd numbers multiplied by 2

1.7 Calculations between ratios (part 1)

In conventional harmony as in numbers-based harmony, it may be useful to know how to calculate the interval between two notes or to perform additions and subtractions of intervals.

1.7.1 Addition

To add two ratios

$$\frac{3}{2} + \frac{4}{3}$$

we must multiply them,

$$\frac{3}{2} \times \frac{4}{3}$$

which is done more specifically by multiplying the numerators between them and the denominators between them.

$$\frac{3 \times 4}{2 \times 3} = \frac{12}{6}$$

The result can be simplified by dividing each of the terms by their greatest common divisor; in this case, 6.

$$\frac{12 \div 6}{6 \div 6} = \frac{2}{1}$$

1.7.2 Difference

To find the interval between two ratios (that is, their difference), we must divide the largest ratio by the smallest,

$$\frac{3}{2} \div \frac{4}{3}$$

which is done by multiplying the first ratio by the inverse of the second.

$$\frac{3}{2} \times \frac{3}{4}$$

$$\frac{3 \times 3}{2 \times 4} = \frac{9}{8}$$

We can also obtain the interval between two ratios by the product of the opposite terms between the smallest ratio and the largest ratio.

$$\frac{4}{3} \boxtimes \frac{3}{2}$$

$$\frac{3 \times 3}{4 \times 2} = \frac{9}{8}$$

1.7.3 Complement interval

A "complement" is the interval that must be added to a ratio to obtain the octave. For example, 4/3 is the complement interval of 3/2, and vice versa, since their sum is 2/1. The complement interval of any denominative ratio is the inversion of that ratio translated into a denominative. For example: to find the complement interval of the ratio 3/2, invert that ratio to obtain 2/3, then translate 2/3 into its denominative 4/3, so that (3/2) + (4/3) = 2/1; to find the complement interval of the ratio 5/4, invert that ratio to obtain 4/5, then translate 4/5 into its denominative 8/5, so that (5/4) + (8/5) = 2/1; etc.

1.8 Modulation factor

The transition from one tonality to another is called "modulation." The interval between two tonalities is the "modulation factor." The modulation factor is denoted by a denominative preceded by a multiplier sign, since, as will be explained in this section, the fundamental of the starting tonality must be multiplied by the modulation factor in order to reach the fundamental of the destination tonality. For example, the modulation factor between the tonalities 4/3* and 8/5* is *6/5.

starting tonality	modulation factor	destination tonality
4/3*	*6/5	8/5*

Table 1.25: Modulation by a *6/5 factor

Just like the notation of a chord in proportions [x:y:z] (see section 1.2.1), the modulation factor makes it possible to create a graph that can be replicated without regard to the specific tonalities involved. The modulation factor also expresses the harmonic common to two tonalities, the numerator corresponding to the harmonic of the starting tonality and the denominator to the harmonic of the destination tonality. For example, the modulation factor *6/5 means that a harmonic of 3-identity in the starting tonality is a harmonic of 5-identity in the destination tonality. The modulation factor therefore also expresses the level of simplicity or complexity of a tonal relationship.

1.8.1 Calculation

Like any interval, the modulation factor is obtained by dividing, in this case by dividing the destination tonality by the starting tonality, so as to retrace the distance travelled. As we saw previously (see section 1.7.2), this operation can be replaced by the product of the opposite terms.

Taking as an example a switch from tonality 4/3* to 8/5*,

$$\frac{4}{3} \rightarrow \frac{8}{5}$$

one must simplify the ratios into prime factors to make the multiplication operation easier,

$$\frac{1}{3} \rightarrow \frac{1}{5}$$

obtain the product of the opposite terms between the starting and destination tonalities

$$\frac{1}{3} \times \frac{1}{5}$$

$$\frac{3 \times 1}{1 \times 5} = \frac{3}{5}$$

and translate the result into a denominative.

$$\frac{3 \times 2}{5} = \frac{6}{5}$$

The proof can be obtained easily by multiplying the starting tonality by the modulation factor (the ratios could have been simplified into prime factors to make this operation easier),

$$\frac{4}{3} \times \frac{6}{5}$$

$$\frac{4 \times 6}{3 \times 5} = \frac{24}{15}$$

then, by translating the result into a denominative.

$$\frac{24 \div 3}{15 \div 3} = \frac{8}{5}$$

The following subsections present, through five demonstrations, various approaches for quickly visualizing the modulation factors throughout the tonal network. But first, here is Table 1.18 again, as we will use it for demonstrations A, B, and C.

1/	5/	7/	
1/1*	5/4*	7/4*	/1
8/5*		7/5*	/5
8/7*	10/7*		/7
16/11*	20/11*	14/11*	/11
16/13*	20/13*	14/13*	/13
32/17*	20/17*	28/17*	/17
32/19*	20/19*	28/19*	/19
32/23*	40/23*	28/23*	/23

Table 1.26: Denominatives for tonalities of orders 1, 5, and 7

1.8.2 Demonstration A

Referring to Table 1.26 above, let us describe the calculation steps to go from tonality 14/11* to tonality 1/1*:

— multiply 14/11* by 11 to arrive at 7/4*, therefore, in other words, put 11 as the numerator of the modulation factor (11/);
— divide 7/4* by 7 to arrive at 1/1*, in other words, put 7 as the denominator of the modulation factor (/7);
— combine the two terms, 11/7, to form the modulation factor *11/7.

starting tonality	modulation factor	destination tonality
14/11*	*11/7	1/1*

Table 1.27: Modulation by a *11/7 factor

[13] Eventually, you could use a similar approach to visualize the interval between any pairs of ratios.

1.8.3 Demonstration B

Still referring to Table 1.26, let us describe the calculations for three modulation options by a *10/7 factor.

Option 1:

— multiply 1/1* by 5 to arrive at 5/4*, in other words, put 5 as the numerator of the modulation factor (5/);
— divide 5/4* by 7 to arrive at 10/7*, in other words, put 7 as the denominator of the modulation factor (/7);
— combine the two terms, 5/7, to get the modulation factor *10/7.

starting tonality	modulation factor	destination tonality
1/1*	*10/7	10/7*

Table 1.28: Modulation by a *10/7 factor (option 1)

Option 2:

— multiply 8/5* by 5 to arrive at 1/1*, in other words, put 5 as the numerator of the modulation factor (5/);
— divide 1/1* by 7 to arrive at 8/7*, in other words, put 7 as the denominator of the modulation factor (/7);
— combine the two terms, 5/7, to get the modulation factor *10/7.

starting tonality	modulation factor	destination tonality
8/5*	*10/7	8/7*

Table 1.29: Modulation by a *10/7 factor (option 2)

Option 3:

— multiply 7/5* by 5 to arrive at 7/4*, in other words, put 5 as the numerator of the modulation factor (5/);
— divide 7/4* by 7 to arrive at 1/1*, in other words, put 7 as the denominator of the modulation factor (/7);
— combine the two terms, 5/7, to get the modulation factor *10/7.

starting tonality	modulation factor	destination tonality
7/5*	*10/7	1/1*

Table 1.30: Modulation by a *10/7 factor (option 3)

1.8.4 Demonstration C

Let us refer one last time to Table 1.26. Let us describe the calculation for the only possible modulation by a *25/16 factor:

— multiply 8/5* by 5 to arrive at 1/1*, in other words, put 5 as the numerator of the modulation factor (5/);
— multiply 1/1* by 5 to arrive at 5/4*, in other words, put 5 as the numerator of the modulation factor (5/);
— no division is required; therefore, place 1 as the denominator of the modulation factor (/1);
— combine the three terms, (5*5)/1, to obtain the modulation factor *25/16.

starting tonality	modulation factor	destination tonality
8/5*	*25/16	5/4*

Table 1.31: Modulation by a *25/16 factor

Conversely, the transition from tonality 5/4* to tonality 8/5* is achieved by the modulation factor *32/25, since 32/25 is the complement interval of 25/16.

starting tonality	modulation factor	destination tonality
5/4*	*32/25	8/5*

Table 1.32: Modulation by a *32/25 factor

1.8.5 Demonstration D

Let us now refer to Table 1.22, which shows the entire tonal network, to visualize the modulation factors between tonalities involving the number 3.

For example, what we need to do to calculate the modulation factor from tonality 1/1* to tonality 10/9* is:

— multiply 1/1* by 5 to arrive at 5/4*, in other words, put 5 as the numerator of the modulation factor (5/);
— divide 5/4* by 3^2 to arrive at 10/9*, in other words, put 3^2 as the denominator of the modulation factor (/3^2);
— combine the two terms, 5/3^2, to form the modulation factor *10/9.

starting tonality	modulation factor	destination tonality
1/1*	*10/9	10/9*

Table 1.33: Modulation by a *10/9 factor

1.8.6 Demonstration E

Still referring to Table 1.22, let us describe the calculation of a modulation by a *24/17 factor:

— multiply 4/3* by 3 to arrive at 1/1*, in other words, put 3 as the numerator of the modulation factor (3/);
— divide 1/1* by 17 to arrive at 32/17*, in other words, put 17 as the denominator of the modulation factor (/17);
— combine the two terms, 3/17, to form the modulation factor *24/17.

starting tonality	modulation factor	destination tonality
1/1*	*24/17	24/17*

Table 1.34: Modulation by a *24/17 factor

In a similar case, here is a quick way to find a generic pair of tonalities associated with a modulation factor:

— the numerator of the starting and destination tonalities is the 1-identity;
— the numerator of the modulation factor is equal to the denominator of the starting tonality;
— the denominator of the modulation factor is equal to the denominator of the destination tonality;
— the tonalities translate into their denominatives.

starting tonality	modulation factor	destination tonality
1	*24/17	1
↓	↙ ↘	↓
1/24	→	1/17
4/3*	→	32/17*

Table 1.35: Pair of tonalities associated with a *24/17 modulation factor

1.9 Calculations between ratios (part 2)

We know that the melodic interval between two harmonics of the same tonality simply corresponds to their relationship in the form of a proportion. For example, for 6- and 7-harmonics, the ascending melodic interval is noted 6 : 7 (+267 c) and the descending melodic interval is noted 7 : 6 (-267 c).

To obtain the melodic interval between two harmonics in different tonalities, divide the destination harmonic by the starting harmonic (to retrace the distance travelled). As we saw previously (see section 1.7.2), this operation can be replaced by the product of the opposite terms.

Taking for example the interval between 7-harmonic of tonality 4/3* and 6-harmonic of tonality 8/5*, one must get the denominative of the starting harmonic

$$4/3^*[7] = \frac{4}{3} \times \frac{7}{1} = \frac{7}{3}, \frac{7}{3 \times 2} = 7/6$$

and the denominative of the destination harmonic

$$8/5^*[6] = \frac{8}{5} \times \frac{6}{1} = \frac{48}{5}, \frac{48 \div 2^3}{5} = 6/5$$

to get the product of the opposite terms between the starting harmonic and the destination harmonic.

$$\frac{7}{6} \times \frac{6}{5} = \frac{6 \times 6}{5 \times 7} = \frac{36}{35}$$

Depending on whether the melodic interval is ascending or descending, it will be noted 35 : 36 (+49 c) or 36 : 35 (-49 c).

If the modulation factor is known, we can deduce the denominatives of the starting and destination pitches from the corresponding harmonics just like we did with pairs of tonalities (see section 1.8.6):

— the starting and destination harmonics remain at the numerator position of the corresponding ratios;
— the numerator of the modulation factor is equal to the denominator of the starting ratio;
— the denominator of the modulation factor is equal to the denominator of the destination ratio.

starting harmonic	modulation factor	destination harmonic
7	*6/5	6

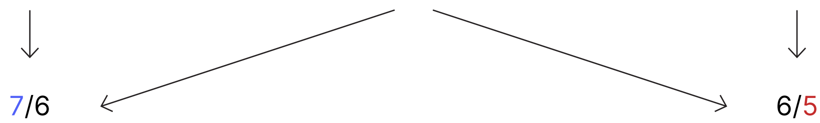
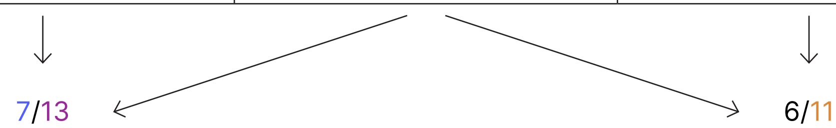


Table 1.36: Pitches associated with a *6/5 modulation factor

Here is another example with the same starting and destination harmonics, this time for a sequence using the *13/11 modulation factor:

starting harmonic	modulation factor	destination harmonic
7	*13/11	6



$$\frac{7}{13} \rightarrow \frac{6}{11}$$

$$\frac{7}{13} \times \frac{6}{11} = \frac{13 \times 6}{7 \times 11} = \frac{78}{77}$$

Table 1.37: Melodic interval associated with a *13/11 modulation factor

Depending on whether the melodic interval is ascending or descending, it will be noted 77 : 78 (+22 c) or 78 : 77 (-22 c).

Conceptually, since all the notes in the tonal network are harmonically related to each other, there is a single fundamental to which they all relate. Going back to our last example, the interval 77 : 78 corresponds to the difference between pitches 14/13 and 12/11, but also to the difference between harmonics 78 and 77 of a theoretical 1-fundamental. Ultimately, the distinction between tonality and tonal network arises from the analytical approach selected.

2. Notation

Chapter 2 describes how to express integer ratios in musical symbols, particularly through the "Helmholtz-Ellis Just Intonation" (HEJI)[1] musical notation system. The HEJI system makes it possible to represent each prime number identity by an accidental symbol of its own, while giving precedence to the 3-identity. Before diving into the study of these accidentals (see sections 2.4 and 2.5), the beginner reader will benefit from the following explanations about the conventional system.

2.1 Deviations in cents

To translate integer ratios into musical notes, we can refer to the conventional Western musical system, use the name of the nearest note and specify the deviation in cents. Let us combine Tables 1.15 and 1.16 presented previously.

note	*C*	*C♯* *D♭*	*D*	*D♯* *E♭*	*E*	*F*	*F♯* *G♭*	*G*	*G♯* *A♭*	*A*	*A♯* *B♭*	*B*	*C*
irrational number	1	$2^{1/12}$	$2^{1/6}$	$2^{1/4}$	$2^{1/3}$	$2^{5/12}$	√2	$2^{7/12}$	$2^{2/3}$	$2^{3/4}$	$2^{5/6}$	$2^{11/12}$	2
cents	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
ratio	1/1	17/16	9/8	19/16	5/4	21/16	23/16	3/2	8/5	27/16	7/4	15/8	2/1
cents	0	105	204	298	386	471	628	702	814	906	969	1088	1200

Table 2.1: Comparison between irrational numbers and ratios

For example, the ratio 9/8 is 204 cents; the nearest conventional note is *D*; 9/8 is 4 cents more than the conventional *D* as measured by a tuner; 9/8 will therefore be noted *D* +4. On the musical staff, the indication "+4" is added above the note *D*.

[1] This system was designed by Marc Sabat and Wolgang von Schweinitz in the early 2000s and updated by Marc Sabat and Thomas Nicholson in 2020. See Nicholson and Sabat, 2020, as well as Nicholson, 2020.

Here is another example: the ratio 5/4 measures 386 cents; the conventional note closest to this ratio is *E*; 5/4 is 14 cents less than the conventional *E* as measured by a tuner; 5/4 will therefore be noted *E* -14. On the musical staff, the indication "-14" is added below the note *D*.

Logically speaking, no cent deviation exceeds ±50. For example, if the intonation of *E* is raised above 50 cents, the nearest reference note becomes *F*. A tuner would read *F* -49 rather than *E* +51.

This method has the benefits of being easy to understand, of allowing the use of a tuner to assess the accuracy of a note played and of limiting the use of too many symbols. However, it has the disadvantages of abstractly quantifying a deviation from a reference that is itself abstract (because it comes from an irrational number), in addition to giving no information about the tonal function of the note played. Also, the tuner which would be used to compare cent deviations will inevitably lack precision, because of the complexity of the sound phenomenon.

Thus, the reflexes of musicians (trained to react to a musical symbol and adjust their intonation in relation to the tonal context that they perceive) are not adequately taken into account by this method, which must therefore be used in conjunction with another.

2.2 Diatonic scale

2.2.1 Diatonism and chromatism

In order to translate integer ratios into musical notes while taking into account the tonal context, we need to deepen our understanding of the conventional Western musical system. Here again are the twelve notes of this system.

note	*C*	*C♯* *D♭*	*D*	*D♯* *E♭*	*E*	*F*	*F♯* *G♭*	*G*	*G♯* *A♭*	*A*	*A♯* *B♭*	*B*	*C*

Table 2.2: The twelve notes of the conventional system

The seven notes in the white columns represent the diatonic scale. The other notes, in the grey columns, represent chromatic alterations of the diatonic scale. The sharp (♯) alters a note by 100 cents upwards; the flat (♭) alters a note by 100 cents downwards. The double sharp (𝄪) and the double flat (𝄫), sometimes necessary, alter a note by 200 cents.

The equivalence of chromatic notes (such as *C♯* and *D♭*) is called "enharmony." Enharmony is necessary to preserve the diatonic relation of the categories of intervals, as we will explain later (see section 2.3).

2.2.2 Ratios multiples of 3

As we mentioned in the introduction, the origins of what we interpret today as the diatonic scale come from ancient Greece. By studying the proportions of a vibrating string, Pythagoras (c. 590-494 B.C.) introduced integer ratios to divide the octave (1∶2) by an interval of a fourth (3∶4) and a fifth (2∶3)[2]. As we know, the number 3 makes it possible to generate notes, while the number 2 makes it possible to arrange these notes within an octave to create a scale.

If we arbitrarily assign the value 1 to the note *F* and multiply its frequency by 3, 3^2, 3^3, etc., we get the following sequence of notes.

note	*F*	*C*	*G*	*D*	*A*	*E*	*B*
factor	*1	*3	$*3^2$	$*3^3$	$*3^4$	$*3^5$	$*3^6$

Table 2.3: Diatonic notes organized in multiples of 3

If we were to extend this sequence towards the treble, it would reproduce identically in sharps, then again in double sharps, since the *B-F♯* interval is equivalent to the *F-C* interval.

note	*F*	*C*	*G*	*D*	*A*	*E*	*B*	*F♯*	*C♯*	*G♯*	*D♯*	*A♯*	*E♯*	*B♯*	*F×*	...
factor	*1	*3	$*3^2$	$*3^3$	$*3^4$	$*3^5$	$*3^6$	$*3^7$	$*3^8$	$*3^9$	$*3^{10}$	$*3^{11}$	$*3^{12}$	$*3^{13}$	$*3^{14}$	...

Table 2.4: Diatonic notes and chromatic sharp transpositions

If we were to extend this sequence towards the bass, it would reproduce identically in flats, then again in double flats, since the *B♭-F* interval is equivalent to the *F-C* interval.

note	...	*B𝄫*	*F♭*	*C♭*	*G♭*	*D♭*	*A♭*	*E♭*	*B♭*	*F*	*C*	*G*	*D*	*A*	*E*	*B*
factor	...	$/3^8$	$/3^7$	$/3^6$	$/3^5$	$/3^4$	$/3^3$	$/3^2$	/3	*1	*3	$*3^2$	$*3^3$	$*3^4$	$*3^5$	$*3^6$

Table 2.5: Diatonic notes and chromatic flat transpositions

[2] Doty, 2002, p. 2.

Let us go back to the original diatonic sequence, this time specifying: the ratio to the starting note; the denominative of this ratio, and; the value of the denominative in cents.

note	*F*	*C*	*G*	*D*	*A*	*E*	*B*
factor	*1	*3	$*3^2$	$*3^3$	$*3^4$	$*3^5$	$*3^6$
ratio	$\frac{1}{1}$	$\frac{3}{1}$	$\frac{9}{1}$	$\frac{27}{1}$	$\frac{81}{1}$	$\frac{243}{1}$	$\frac{729}{1}$
denominative	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{9}{8}$	$\frac{27}{16}$	$\frac{81}{80}$	$\frac{243}{128}$	$\frac{729}{512}$
cents	0	702	204	906	408	1110	612

Table 2.6: Values of diatonic notes multiples of 3, for *F* = 1/1

If we arbitrarily decide that the value of the note *F* is 1/3, rather than 1, we get the following values.

note	*F*	*C*	*G*	*D*	*A*	*E*	*B*
factor	/3	*1	*3	$*3^2$	$*3^3$	$*3^4$	$*3^5$
ratio	$\frac{1}{3}$	$\frac{1}{1}$	$\frac{3}{1}$	$\frac{9}{1}$	$\frac{27}{1}$	$\frac{81}{1}$	$\frac{243}{1}$
denominative	$\frac{4}{3}$	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{9}{8}$	$\frac{27}{16}$	$\frac{81}{80}$	$\frac{243}{128}$
cents	498	0	702	204	906	408	1110

Table 2.7: Values of diatonic notes multiples of 3, for *C* = 1/1

If we organize this last sequence in ascending order of cents, we find the order of notes common today.

note	*C*	*D*	*E*	*F*	*G*	*A*	*B*	*C*
ratio	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	$\frac{2}{1}$
cents	0	204	408	498	702	906	1110	1200

Table 2.8: Scaling of diatonic notes multiples of 3, starting with *C*

2.2.3 Ratios multiples of 5

In his work entitled *Harmonics*, Claudius Ptolemy (circa 90-168 AD) described what he calls the “tense diatonic” mode using proportions that integrate the number 5, in addition to numbers 2 and 3. To present this mode, let us again arbitrarily attribute the ratio 1/1 to the note *C*. We find that the ratios of the notes *E*, *A* and *B* in the Ptolemaic mode differ from the same in the Pythagorean mode.

note	*C*	*D*	*E*	*F*	*G*	*A*	*B*	*C*
ratio	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{2}{1}$
cents	0	204	386	498	702	884	1088	1200

Table 2.9: Scaling of diatonic notes multiples of 3 and 5, starting with *C*

To better study this mode,

— let us first have the information about each element inside a cell, as follows (dev. stands for deviation);

ratio	note
cents	dev.

Table 2.10: Organizing the information in a table cell

— then, let us arrange the multiples of 3 horizontally and the multiples of 5 vertically, as follows.

5	$\frac{5}{3}$ *A* 884 -16	$\frac{5}{4}$ *E* 386 -14	$\frac{15}{8}$ *B* 1088 -12	
1	$\frac{4}{3}$ *F* 498 -2	$\frac{1}{1}$ *C* 0 +0	$\frac{3}{2}$ *G* 702 +2	$\frac{9}{8}$ *D* 204 +4

Table 2.11: Organizing diatonic notes according to multiples of 3 and 5

A posteriori, we can see that this mode is based on what is called, in the conventional system, the “major triad,” that is to say, in terms of harmonic relations, a triad composed of the identities 1-3-5:

— *C*-*E*-*G* triad;

5	5/4 *E* 386 -14	
1	1/1 *C* 0 +0	3/2 *G* 702 +2

Table 2.12: *C*-*E*-*G* triad

— *F*-*A*-*C* triad (i.e. the *C*-*E*-*G* triad multiplied by the 3-subharmonic);

5	5/3 *A* 884 -16	
1	4/3 *F* 498 -2	1/1 *C* 0 +0

Table 2.13: *F*-*A*-*C* triad

— *G*-*B*-*D* triad (i.e. the *C*-*E*-*G* triad multiplied by the 3-harmonic).

5	15/8 *B* 1088 -12	
1	3/2 *G* 702 +2	9/8 *D* 204 +4

Table 2.14: *G*-*B*-*D* triad

2.2.4 Differences in intonation

Let us combine the ratios multiples of 3 and 5 in the same table. For the notes *A*, *E* and *B*, there are two different intonations possible, depending on whether they correspond to a 3-identity or a 5-identity.

5	5/3 *A* 884 -16	5/4 *E* 386 -14	15/8 *B* 1088 -12				
1	4/3 *F* 498 -2	1/1 *C* 0 +0	3/2 *G* 702 +2	9/8 *D* 204 +4	27/16 *A* 906 +6	81/64 *E* 408 +8	243/128 *B* 1110 +10

Table 2.15: Differences in intonation between notes *A*, *E* and *B* according to their identity

These differences recur as more notes are added. For example, *F*♯ as a 5-identity of *D* would differ from *F*♯ as a 3-identity of *B*, etc.

5	5/3 *A* 884 -16	5/4 *E* 386 -14	15/8 *B* 1088 -12	45/32 ♯*F* 590 -10				
1	4/3 *F* 498 -2	1/1 *C* 0 +0	3/2 *G* 702 +2	9/8 *D* 204 +4	27/16 *A* 906 +6	81/64 *E* 408 +8	243/128 *B* 1110 +10	729/512 ♯*F* 612 +12

Table 2.16: Differences in intonation for the note *F*♯ depending on its identity

From the end of the 18th century to the middle of the 19th century, the division of the octave into twelve equal parts gradually established itself in the West as a solution to disregard these differences in intonation. Thus, in the Western musical system, each note corresponds to a single intonation:

— a single value of 900 cents is used for both *A* = 5/3 (884 c) and *A* = 27/16 (906 c);
— a single value of 400 cents is used for both *E* = 5/4 (386 c) and *E* = 81/64 (408 c);
— a single value of 1100 cents is used for both *B* = 15/8 (1088 c) and *B* = 243/128 (1110 c);
— etc.

According to that system, the intonation of 5-identities is slightly raised, while that of 3-identities is lowered very slightly. The symmetry of the system obtained makes it possible to transpose the major triad on each of the twelve notes while always preserving exactly the same interval widths, without causing too great a deviation from the implied ratio of integers. This quasi-equivalence between notes generated by the prime numbers 2, 3 and 5 is due to random mathematics: theoretically, 3 being a prime number, multiplying a frequency by 3 to infinity generates an infinity of different notes. However, multiplying a frequency by 3 twelve times (3^{12}) produces twelve notes, the last of which is very close to one of the octaves of the starting frequency. Indeed, multiplying a frequency by 3 twelve times produces an interval barely greater than if we multiply the same frequency by 2 nineteen times, as illustrated by the following mathematical calculations:

$1*3^{12} = 531\,441 \approx 1*2^{19} = 524\,288$

or

$(3/2)^{12} = 129{,}746... \approx (2/1)^{7} = 128$

On the other hand, multiplying a frequency by 3 four times (3^4) produces a note that approaches the 5-identity of the starting frequency. Indeed, multiplying a frequency by 3/2 four times produces an interval a little wider than if we multiply the same frequency by 5, as illustrated by the following mathematical calculations:

$1*(3/2)^4 = 5{,}0625 \approx 1*5 = 5$

or

$3^4 = 81 \approx 5*2^4 = 80$

Logically, dividing the octave into twelve equal parts generates twelve notes having between them: twelve perfect intervals of 1 : 2, twelve intervals barely smaller than 2 : 3; twelve intervals slightly larger than 4 : 5.

The notation system we are using differs from this approach. We refer to ratios multiples of 3 and their traditional accidentals to generate the notes; then, we use specific accidentals to specify the intonation of these same notes according to their identity (5, 7, 11, etc.). To understand which note to choose according to the identity, we must once again return to the study of the diatonic scale, this time to understand the denomination of its intervals.

2.3 Diatonic intervals

2.3.1 Names of intervals

Each of the seven notes on the diatonic scale is assigned a degree. The number of degrees involved in a given interval determines the name of that interval. The scale is named after its first degree. Thus, for the diatonic scale of *C*, we see that

— the *C*-*D* interval implicates two degrees, so is called a "second" and measures 200 cents (200 c - 0 c = 200 c);
— the *C*-*E* interval implicates three degrees, is therefore called a "third," and measures 400 cents (400 c - 0 c = 400 c);
— etc.

note	*C*	*D*	*E*	*F*	*G*	*A*	*B*	*C*
degree	1	2	3	4	5	6	7	8
interval	prime	second	third	fourth	fifth	sixth	seventh	octave
cents	0	200	400	500	700	900	1100	1200

Table 2.17: Categories of diatonic intervals according to the degree of the scale

However,

— the *E*-*F* interval implicates two degrees and is thus also called a "second," yet measures 100 cents (500 c - 400 c = 100 c) ;
— the *E*-*G* interval implicates three degrees, is also called a "third," and yet measures 300 cents (700 c - 400 c = 300 c);
— etc.

Intervals with the same name differ in width. This occurs in several places in the scale and generates a variety of intervals of different qualities. Here is, mirrored, the Table 2.18 of the intervals of the conventional system that we use.

The intervals are reversed as follows: prime/octave; second/seventh; third/sixth; fourth/fifth. Qualities are reversed according to the same principle: major/minor; augmented/diminished. Therefore, the inversion of a minor third (*E*-*G*) corresponds to a major sixth (*G*-*E*); the inversion of an augmented fourth (*C*-*F*♯) corresponds to a diminished fifth (*F*♯-*C*); etc.

<table>
<tr><th>interval</th><th>quality</th><th>cents</th></tr>
<tr><td rowspan="2">prime</td><td>perfect</td><td>0</td></tr>
<tr><td>augmented</td><td rowspan="2">100</td></tr>
<tr><td rowspan="2">second</td><td>minor</td></tr>
<tr><td>major</td><td>200</td></tr>
<tr><td rowspan="2">third</td><td>minor</td><td>300</td></tr>
<tr><td>major</td><td rowspan="2">400</td></tr>
<tr><td rowspan="3">fourth</td><td>diminished</td></tr>
<tr><td>perfect</td><td>500</td></tr>
<tr><td>augmented</td><td rowspan="2">600</td></tr>
<tr><td rowspan="3">fifth</td><td>diminished</td></tr>
<tr><td>perfect</td><td>700</td></tr>
<tr><td>augmented</td><td rowspan="2">800</td></tr>
<tr><td rowspan="2">sixth</td><td>minor</td></tr>
<tr><td>major</td><td>900</td></tr>
<tr><td rowspan="2">seventh</td><td>minor</td><td>1000</td></tr>
<tr><td>major</td><td rowspan="2">1100</td></tr>
<tr><td rowspan="2">octave</td><td>diminished</td></tr>
<tr><td></td><td>1200</td></tr>
</table>

<table>
<tr><th>cents</th><th>quality</th><th>interval</th></tr>
<tr><td>1200</td><td>perfect</td><td rowspan="2">octave</td></tr>
<tr><td rowspan="2">1100</td><td>diminished</td></tr>
<tr><td>major</td><td rowspan="2">seventh</td></tr>
<tr><td>1000</td><td>minor</td></tr>
<tr><td>900</td><td>major</td><td rowspan="2">sixth</td></tr>
<tr><td rowspan="2">800</td><td>minor</td></tr>
<tr><td>augmented</td><td rowspan="3">fifth</td></tr>
<tr><td>700</td><td>perfect</td></tr>
<tr><td rowspan="2">600</td><td>diminished</td></tr>
<tr><td>augmented</td><td rowspan="3">fourth</td></tr>
<tr><td>500</td><td>perfect</td></tr>
<tr><td rowspan="2">400</td><td>diminished</td></tr>
<tr><td>major</td><td rowspan="2">third</td></tr>
<tr><td>300</td><td>minor</td></tr>
<tr><td>200</td><td>major</td><td rowspan="2">second</td></tr>
<tr><td rowspan="2">100</td><td>minor</td></tr>
<tr><td>augmented</td><td rowspan="2">prime</td></tr>
<tr><td>0</td><td>perfect</td></tr>
</table>

Table 2.18: List of diatonic intervals and their qualities

By keeping the same starting note, inverting the interval is equivalent to changing its direction (ascending or descending). So, from *E*, the ascending minor third interval is *E-G* and the descending minor third (or ascending major sixth) interval is *E-C♯*; from *C*, the ascending augmented fourth interval is *C-F♯* and the descending augmented fourth (or ascending diminished fifth) interval is *C-G♯*; etc.

Any interval that exceeds an octave is called a “compound” interval. For example, 1:4 is a “compound octave.” The same principle applies to seconds, thirds, etc.

2.3.2 Demonstrations

Let us take another look at Table 1.15:

note	*C*	*C♯* *D♭*	*D*	*D♯* *E♭*	*E*	*F*	*F♯* *G♭*	*G*	*G♯* *A♭*	*A*	*A♯* *B♭*	*B*	*C*
cents	0	100	200	300	400	500	600	700	800	900	1000	1100	1200

Table 2.19: The twelve notes of the conventional system and their values in cents

For example, let us find out which note represents the minor third of *D♯*. To do this, let us involve the three degrees of the diatonic scale concerned: *D*, *E*, *F*. We know that the third of *D* has the name *F*, regardless of the accidentals involved. To ensure that *F* is properly altered in relation to *D* to form a minor third, add the cents value of a minor third (300 c) to the cents value of *D* (300 c): 300 c + 300 c = 600 c. The 600 cents value is *F#* or *G♭*. We know that the correct notation is *F♯*. The minor third of *D♯* is *F♯*.

For another example, let us find out of which note *G* is the major third. To do this, subtract the cents value of *G* (700 c) from the cents value of a major third (400 c): 700 c - 400 c = 300 c. The 300 cents value corresponds to the note *E♭*. *G* is therefore the major third of *E♭*. To be sure, we see that the interval does indeed involve three diatonic degrees (*E*, *F*, *G*).

2.3.3 Identity interval

In the notation system we adopted, each prime number identity corresponds to a specific diatonic interval.

prime number identity	diatonic interval to alter
23	augmented fourth
19	minor third
17	augmented prime
13	major sixth
11	perfect fourth
7	minor seventh
5	major third
3	perfect fifth

Table 2.20: Interval categories based on identities

For the sake of simplicity, let us always consider that the interval from fundamental to harmonic is ascending, while the interval from harmonic to fundamental is descending. For example, the interval from 1-fundamental to 5-harmonic corresponds to an ascending major third, 1 : 5, or 4 : 5; the interval from 1-harmonic to 1/5-fundamental corresponds to a descending major third, 5 : 1, or 5 : 4.

After having determined the diatonic interval and its corresponding note according to the identity to be represented, an accidental comes in to modify its intonation.

2.4 Accidentals

The following table summarizes all of the accidental symbols we use and their values.

identity	symbol					ratio		cents
49/	⌊⌊𝄫	⌊⌊♭	⌊⌊	⌊⌊♯	⌊⌊𝄪	$(64:63)^2$	$(3^2*7/1)^2$	-55
25/	𝄫⇊	♭⇊	♮⇊	♯⇊	𝄪⇊	$(81:80)^2$	$(3^4/5)^2$	-43
23/	↑𝄫	↑♭	↑	↑♯	↑𝄪	729:736	$3^6/23$	+17
19/	⁄𝄫	⁄♭	⁄	⁄♯	⁄𝄪	512:513	$1/3^3*19$	+3
17/	≷𝄫	≷♭	≷	≷♯	≷𝄪	2187:2176	$3^7/2^7*17$	-9
13/	◁𝄫	◁♭	◁	◁♯	◁𝄪	27:26	$3^3/13$	-65
11/	⫲𝄫	⫲♭	⫲	⫲♯	⫲𝄪	33:32	$3*11/1$	+53
7/	⌊𝄫	⌊♭	⌊	⌊♯	⌊𝄪	64:63	$1/3^2*7$	-27
5/	𝄫↓	♭↓	♮↓	♯↓	𝄪↓	81:80	$3^4/5$	-22
3/	𝄫	♭	♮	♯	𝄪	2048:2187	$2^{11}/3^7$	+114
/3						2187:2048	$3^7/2^{11}$	-114
/5	𝄫↑	♭↑	♮↑	♯↑	𝄪↑	80:81	$5/3^4$	+22
/7	⌈𝄫	⌈♭	⌈	⌈♯	⌈𝄪	63:64	$3^2*7/1$	+27
/11	◁𝄫	◁♭	◁	◁♯	◁𝄪	33:32	$3*11/1$	-53
/13	⫲𝄫	⫲♭	⫲	⫲♯	⫲𝄪	26:27	$13/3^3$	+65
/17	≶𝄫	≶♭	≶	≶♯	≶𝄪	2187:2176	$3^7/2^7*17$	+9
/19	\𝄫	\♭	\	\♯	\𝄪	513:512	$3^3*19/1$	-3
/23	↓𝄫	↓♭	↓	↓♯	↓𝄪	736:729	$23/3^6$	-17

Table 2.21: Summary table of 23-limit accidental symbols of the HEJI system

Before we discuss each of these accidentals in the following subsections, here are some general considerations about the notation system used in the *Treatise*.

The previous table shows that each identity has specific harmonic and subharmonic accidental symbols. The flat and sharp accidentals become “3-limit” accidentals by only referring to ratios that are multiples of 3. Indeed, as we have seen by studying the diatonic and chromatic scales, the number 3 is sufficient to generate the notes and the conventional accidentals; it is therefore logical that 5-limit to 23-limit accidentals would only modify their intonation.

The notes generated by the multiples of 3 have a peculiarity, compared to the twelve notes of the conventional system: the more they “go up” towards the sharps, the greater the positive cents deviation[3]; the more they “go down” towards the flats, the greater the negative cents deviation. To see this, here is an example with respect to the generating pitch *A* = 1/1:

note	...	*A*♭	*E*♭	*B*♭	*F*	*C*	*G*	*D*	*A*	*E*	*B*	*F*♯	*C*♯	*G*♯	*D*♯	*A*♯	...
denominative	...	4096/2187	1024/729	256/243	128/81	32/27	16/9	4/3	1/1	3/2	9/8	27/16	81/64	243/128	729/512	2187/2048	...
cents	...	1086	588	90	792	294	996	498	0	702	204	906	408	1110	612	114	...
deviation	...	-14	-12	-10	-8	-6	-4	-2	+0	+2	+4	+6	+8	+10	+12	+14	...

Table 2.22: Increasing cents deviations for notes multiples of 3

So much so that enharmony in the conventional sense no longer exists. This can easily be seen by comparing, for example, the notes *G*♯ +10 and *A*♭ -14. Consequently, each pitch has a unique notation.

[3] Cents deviations always refer to the division of the octave into twelve equal parts; therefore, they always apply to the altered note as seen in that system (and regardless of the 5-limit to 23-limit accidentals which may then be added). Depending on their respective logics, it may happen that the note altered by a symbol differs from the equivalent note in cents deviation. In this case, the deviation in cents is accompanied by the name of the note as it would appear on a tuner.

In the following subsections, we will use a specific table template (see section 2.2.3) which will also be used to present all the notes in a tonal network in chapters 4 and 5. Here is a reminder of how that template is built.

First, the information about each harmonic is arranged inside a cell (dev. stands for deviation).

ratio	note
cents	dev.

Table 2.23: Organizing the information in a table cell

Second, the cells of harmonics multiples of 3 are ordered from left to right, as in the example below.

$\frac{4}{3}$	♮D	$\frac{1}{1}$	♮A	$\frac{3}{2}$	♮E
498	-2	0	+0	702	+2

Table 2.24: Cells organized from left to right in multiples of 3

Third, the cells for the prime harmonics multiples of 5 to 23 are tiled from bottom to top, as in the example below.

5	
1	

Table 2.25: Cells organized from bottom to top in multiples of 5

2.4.1 The 3-identity

The 3-limit accidentals are the flat, sharp, and natural symbols. The natural (♮) is the "neutral" accidental used to specify that the note is neither flat nor sharp.

identity	symbol				
3	𝄫	♭	♮	♯	𝄪

Table 2.26: 3-identity accidental symbols

The flat and sharp accidentals, when they relate to multiples of 3 and no longer to irrational numbers, have a value of 114 cents, rather than 100 cents. For example, the difference in intonation between the two notes *A* +0 = 1/1 (0 c) and *A*♯ +14 = 2187/2048 (114 c) corresponds to the interval 2048 : 2187 (+114 c), the factors of which are $2^{11}/3^7$.

The sharp alteration raises the intonation by 114 cents.

1	1/1	♮A	3/2	♮E	9/8	♮B	27/16	♯F	81/64	♯C	243/128	♯G	729/512	♯D	2187/2048	♯A
	0	+0	702	+2	204	+4	906	+6	408	+8	1110	+10	612	+12	114	+14

Table 2.27: Sharp accidental (3-limit)

By reversing the direction of the comparison, the difference in intonation between the two notes *A* +0 = 1/1 (0 c) and *A*♭ -14 = 4096/2187 (1086 c) corresponds to the interval 2187 : 2048 (-114 c), whose factors are $3^7/2^{11}$.

The flat accidental lowers the intonation by 114 cents.

1	4096/2187	♭A	1024/729	♭E	256/243	♭B	128/81	♮F	32/27	♮C	16/3	♮G	4/3	♮D	1/1	♮A
	1086	-14	588	-12	90	-10	792	-8	294	-6	996	-4	498	-2	0	+0

Table 2.28: Flat accidental (3-limit)

2.4.2 The 5-identity

The 5-limit accidental symbol is an arrow pointing up or down attached to a 3-limit accidental.

identity	symbol				
5/	𝄫↓	♭↓	♮↓	♯↓	𝄪↓
/5	𝄫↑	♭↑	♮↑	♯↑	𝄪↑

Table 2.29: 5-identity accidental symbols

The diatonic interval corresponding to the 5-identity is the major third. For a note that would be *A*, the ascending major third is *C♯*. The difference in intonation between the two notes *C♯* +8 = 81/64 (408 c) 3-limit and *C♯* -14 = 5/4 (386 c) 5-limit corresponds to the interval 81 : 80 (-22 c), the factors of which are $3^4/5$.

The accidental of the 5-harmonic lowers the intonation of the 3-limit accidental by 22 cents.

5	5/4 ♯↓C 386 -14	15/8 ♯↓G 1088 -12	45/32 ♯↓D 590 -10		
1	1/1 ♮A 0 +0	3/2 ♮E 702 +2	9/8 ♮B 204 +4	27/16 ♯F 204 +6	81/64 ♯C 204 +8

Table 2.30: 5-harmonic accidental

Reversing the interval, the descending major third of *A* is *F*. The difference in intonation between the two notes *F* -8 = 128/81 (792 c) 3-limit and *F* +14 = 8/5 (814 c) 5-limit corresponds to the interval 80 : 81 (+22 c), whose factors are $5/3^4$. The 5-subharmonic accidental increases the intonation of the 3-limit accidental by 22 cents.

5	128/81 ♮F 792 -8	32/27 ♮C 294 -6	16/9 ♮G 996 -4	4/3 ♮D 498 -2	1/1 ♮A 0 +0
1	512/405 ♭↑D 406 +6	256/135 ♭↑A 1108 +8	64/45 ♭↑E 610 +10	16/15 ♭↑B 112 +12	8/5 ♮↑F 814 +14

Table 2.31: 5-subharmonic accidental

In the same way as 3-limit accidentals (such as the double sharp or the double flat), 5-limit accidentals can be added together.

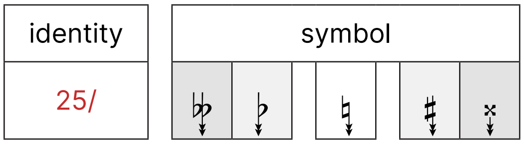


Table 2.32: Addition of the 5-harmonic accidental symbol with itself

2.4.3 The 7-identity

The 7-limit accidental symbol is similar to the shape of the digit “7” facing up or down.

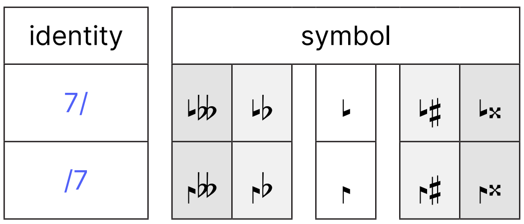


Table 2.33: 7-identity accidental symbols

The diatonic interval corresponding to the 7-identity is the minor seventh. For a note that would be *A*, the ascending minor seventh is *G*. The difference in intonation between the two notes *G* -4 = 16/9 (996 c) 3-limit and *G* -31 = 7/4 (969 c) 7-limit corresponds to the interval 64:63 (-27 c), whose factors are $1/3^2*7$.

The accidental of the 7-harmonic lowers the intonation of the 3-limit accidental by 27 cents.

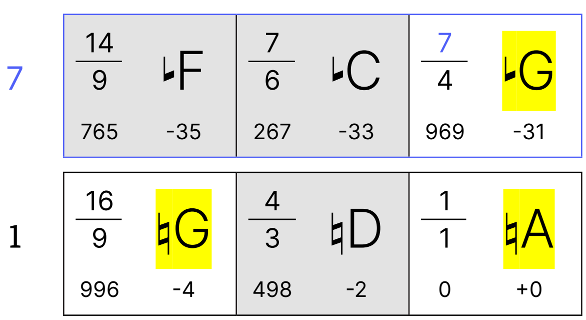


Table 2.34: 7-harmonic accidental

Reversing the interval, the descending minor seventh of *A* is *B*. The difference in intonation between the two notes *B* +4 = 9/8 (204 c) 3-limit and *B* +31 = 8/7 (231 c) 7-limit corresponds to the interval 63 : 64 (+27 c), whose factors are 3^2*7/1.

The 7-subharmonic accidental increases the intonation of the 3-limit accidental by 27 cents.

7	1/1	♮A	3/2	♮E	9/8	♮B
	0	+0	702	+2	204	+4
1	8/7	B	12/7	♯F	9/7	♯C
	231	+31	933	+33	435	+35

Table 2.35: 7-subharmonic accidental

In the same way as 3-limit and 5-limit accidentals, 7-limit accidentals can be added together.

identity	symbol				
49/	♭♭	♭	(empty)	♯	𝄪

Table 2.36: Addition of the 7-harmonic accidental symbol with itself

2.4.4 The 11-identity

The 11-limit accidental symbols are similar to an incomplete sharp or inverted flat.

identity	symbol				
11/	♭♭	♭	(empty)	♯	𝄪
/11	♭♭	♭	(empty)	♯	𝄪

Table 2.37: 11-identity accidental symbols

The diatonic interval corresponding to the 11-identity is the perfect fourth. For a note that would be *A*, the ascending perfect fourth is *D*. The difference in intonation between the two notes *D* -2 = 4/3 (498 c) 3-limit and *D* +51 = 11/8 (551 c) 11-limit corresponds to the interval 32 : 33 (+53 c), whose factors are 1/3*11.

The accidental of the 11-harmonic raises the intonation of the 3-limit accidental by 53 cents.

11	11/6	G	11/8	D
	1049	+49	551	D♯-49
1	4/3	♮D	1/1	♮A
	498	-2	0	+0

Table 2.38: 11-harmonic accidental

Reversing the interval, the descending perfect fourth of *A* is *E*. The difference in intonation between the two notes *E* +2 = 3/2 (702 c) 3-limit and *E* -51 = 16/11 (649 c) 11-limit corresponds to the interval 33 : 32 (-53 c), whose factors are 3*11/1.

The 11-subharmonic accidental lowers the intonation of the 3-limit accidental by 53 cents.

11	1/1	♮A	3/2	♮E
	0	+0	702	+2
1	16/11	E	12/11	B
	649	E♭+49	151	-49

Table 2.39: 11-subharmonic accidental

2.4.5 The 13-identity

The 13-limit accidental symbols are similar to those for the 11-limit with an extra vertical line added to represent a larger value.

identity	symbol				
13/	♭♭	♭		♯	𝄪
/13	♭♭	♭		♯	𝄪

Table 2.40: 13-identity accidental symbols

The diatonic interval corresponding to the 13-identity is the major sixth. For a note that would be *A*, the ascending major sixth is *F*. The difference in intonation between the two notes *F♯* +6 = 27/16 (906 c) 3-limit and *F♯* -59 = 13/8 (841 c) 13-limit corresponds to the interval 27 : 26 (-65 c), whose factors are $3^3/13$.

The 13-harmonic accidental lowers the intonation of the 3-limit accidental by 65 cents.

13	13/8	♯F	39/32	♯C	117/64	♯G		
	841	F+41	342	C+42	1044	G+44		
1	1/1	♮A	3/2	♮E	9/8	♮B	27/16	♯F
	0	+0	702	+2	204	+4	906	+6

Table 2.41: 13-harmonic accidental

Reversing the interval, the descending major sixth of *A* is *C*. The difference in intonation between the two notes *C* -6 = 32/27 (294 c) 3-limit and *C* +59 = 16/13 (359 c) 13-limit corresponds to the interval 26 : 27 (+65 c), whose factors are $13/3^3$.

The 13-subharmonic accidental raises the intonation of the 3-limit accidental by 65 cents.

13	32/27	♮C	16/9	♮G	4/3	♮D	1/1	♮A
	294	-6	996	-4	498	-2	0	+0
1	512/351	♯♭E	128/117	♯♭B	64/39	♯F	16/13	♯C
	654	E-46	156	B-44	858	F#-42	359	C#-41

Table 2.42: 13-subharmonic accidental

2.4.6 The 17-identity

The 17-limit accidental symbol consists of two short oblique lines facing up or down.

identité	symbole				
17/	⋱♭♭	⋱♭	⋱	⋱♯	⋱𝄪
/17	⋰♭♭	⋰♭	⋰	⋰♯	⋰𝄪

Tableau 2.43 : Symboles d'altération de l'identité 17

The diatonic interval corresponding to the 17-identity is the augmented prime. For a note that would be *A*, the ascending augmented prime is *A♯*. The difference in intonation between the two notes *A♯* +14 = 2187/2048 (114 c) 3-limit and *A♯* +5 = 17/16 (105 c) 17-limit is the interval 2187 : 2176 (-9 c), whose factors are $3^7/17$.

The 17-harmonic accidental lowers the intonation of the 3-limit accidental by 9 cents.

17	17/16 ⋱♯A 105 +5	51/32 ⋱♯E 807 +7	153/128 ⋱♯B 309 +9					
1	1/1 ♮A 0 +0	3/2 ♮E 702 +2	9/8 ♮B 204 +4	27/16 ♯F 906 +6	81/64 ♯C 408 +8	243/128 ♯G 1110 +10	729/512 ♯D 612 +12	2187/2048 ♯A 114 +14

Table 2.44: 17-harmonic accidental

By reversing the interval, the descending augmented prime of *A* is *A♭*. The difference in intonation between the two notes *A♭* -14 = 4096/2187 (1086 c) 3-limit and *A♭* -5 = 32/17 (1095 c) limite-17 correspond à l'intervalle 2176 : 2187 (+9 c), whose factors are $17/3^7$.

The 17-subharmonic accidental raises the intonation of the 3-limit accidental by 9 cents.

17	4096/2187 ♭A 1086 -14	1024/729 ♭E 588 -12	256/243 ♭B 90 -10	128/81 ♮F 792 -8	32/27 ♮C 294 -6	16/3 ♮G 996 -4	4/3 ♮D 498 -2	1/1 ♮A 0 +0
1					512/459 ⋰♭C 189 B-11	256/153 ⋰♭G 891 -9	64/51 ⋰♭D 393 -7	32/17 ⋰♭A 1095 -5

Table 2.45: 17-subharmonic accidental

2.4.7 The 19-identity

The 19-limit accidental symbols are similar to those for the 17-limit minus one oblique line to represent a smaller value.

identity	symbol				
19/	/𝄫	/♭	/	/♯	/𝄪
/19	\𝄫	\♭	\	\♯	\𝄪

Table 2.46: 19-identity accidental symbols

The diatonic interval corresponding to the 19-identity is the minor third. For a note that would be *A*, the ascending minor third is *C*. The difference in intonation between the two notes *C* -6 = 32/27 (294 c) 3-limit and *C* -2 = 19/16 (298 c) 19-limit corresponds to the interval 512 : 513 (+3 c), whose factors are $1/3^3*19$.

The 19-harmonic accidental raises the intonation of the 3-limit accidental by 3 cents[4].

19	38/27	/♭E	19/18	/♭B	19/12	/F	19/16	/C
	592	-8	94	-6	796	-4	298	-2
1	32/27	♮C	16/9	♮G	4/3	♮D	1/1	♮A
	294	-6	996	-4	498	-2	0	+0

Table 2.47: 19-harmonic accidental

Reversing the interval, the descending minor third of *A* is *F*♯. The difference in intonation between the two notes *F*♯ +6 = 27/16 (906 c) 3-limit and *F*♯ +2 = 32/19 (902 c) 19-limit corresponds to the interval 513 : 512 (-3 c), whose factors are $3^3*19/1$.

[4] The difference between 32/27 (294 c) and 19/16 (298 c) appears to be 4 cents, but it does correspond to the interval 512 : 513, which is 3 cents. This is due to the rounding of the values, which are, more precisely: 32/27 (294.1 c), 19/16 (297.5 c) and 512 : 513 (3.4 c).

The 19-subharmonic accidental lowers the intonation of the 3-limit accidental by 3 cents.

	ratio	note	cents	offset	ratio	note	cents	offset	ratio	note	cents	offset	ratio	note	cents	offset
19	1/1	♮A	0	+0	3/2	♮E	702	+2	9/8	♮B	204	+4	27/16	♯F	906	+6
1	32/19	\♯F	902	+2	24/19	\♯C	404	+4	36/19	\♯G	1106	+6	27/19	\♯D	608	+8

Table 2.48: 19-subharmonic accidental

2.4.8 The 23-identity

The 23-limit accidental symbol consists of an arrow pointing up or down.

identity	symbol				
23/	↑♭♭	↑♭	↑	↑♯	↑𝄪
/23	↓♭♭	↓♭	↓	↓♯	↓𝄪

Table 2.49: 23-identity accidental symbols

The diatonic interval corresponding to the 23-identity is the augmented fourth. For a note that would be *A*, the ascending augmented fourth is *D♯*. The difference in intonation between the two notes *D♯* +12 = 729/512 (612 c) 3-limit and *D♯* +28 = 23/16 (628 c) 23-limit corresponds to the interval 729 : 736 (+17 c), whose factors are 3^6/23.

The 23-harmonic accidental raises the intonation of the 3-limit accidental by 17 cents[5].

	ratio	note	cents	offset	ratio	note	cents	offset	ratio	note	cents	offset	ratio	note	cents	offset	ratio	note	cents	offset	ratio	note	cents	offset	ratio	note	cents	offset
23	23/16	↑♯D	628	+28	69/64	↑♯A	130	+30	207/128	↑♯E	832	+32																
1	1/1	♮A	0	+0	3/2	♮E	702	+2	9/8	♮B	204	+4	27/16	♯F	906	+6	81/64	♯C	408	+8	243/128	♯G	1110	+10	729/512	♯D	612	+12

Table 2.50: 23-harmonic accidental

[5] The difference between 729/512 (612 c) and 23/16 (628 c) appears to be 16 cents, but it does correspond to the interval 729 : 736, which is 17 cents. This is due to the rounding of the values, which are, more precisely: 729/512 (611.7 c), 23/16 (628.3 c) and 729 : 736 (16.5 c).

By reversing the interval, the descending augmented fourth of *A* is *E*♭. The difference in intonation between the two notes *E*♭ -12 = 1024/729 (588 c) 3-limit and *E*♭ -28 = 32/23 (572 c) 23-limit corresponds to the interval 736 : 729 (-17 c), whose factors are $23/3^6$.

The 23-subharmonic accidental lowers the intonation of the 3-limit accidental by 17 cents.

23	1024/729 ♭E 588 -12	256/243 ♭B 90 -10	128/81 ♮F 792 -8	32/27 ♮C 294 -6	16/3 ♮G 996 -4	4/3 ♮D 498 -2	1/1 ♮A 0 +0
1				1024/621 ♭G 866 -34	256/207 ↓♭D 368 -32	128/63 ↓♭A 1070 -7	32/23 ↓♭E 572 -28

Table 2.51: 23-subharmonic accidental

2.5 Combined accidentals

2.5.1 Tonal function

In addition to specifying the intonation with respect to an absolute reference (i.e. the chosen frequency of the tuning fork for the generating pitch 1/1), accidentals make it possible to trace the tonal functions involved. For example, let's see how to represent the pitch 20/13 by its alteration symbol. The harmonic's identity is 5, while the subharmonic's identity is 13. The symbols associated with these identities are as follows:

identity	symbol
5/	♮↓
/13	♯

Table 2.52: Accidental symbols of 5-harmonic and 13-subharmonic

Their combination gives us the following symbol:

ratio	symbol
20/13	♯♮↓

Table 2.53: Accidental symbol for ratio 20/13

Symbols are always organized this way, from left to right, from the highest to the lowest identity represented, in this case: 13, 5. This convention only serves to standardize the notation: theoretically, inverting the symbols does not change the intonation (in the same way that changing the order of additions and subtractions does not change the result of an equation).

Let us now consider that the ratio 20/13 corresponds to a fundamental and that we seek to note its 7-identity, i.e. 20/13*[7]. The harmonics' identities are 5 and 7 while the subharmonic's identity is 13.

identity	symbol
7/	˪
5/	♮↓
/13	ǂ

Table 2.54: Accidental symbols of harmonics 5 and 7 as well as subharmonic 13

The combination of these tonal functions results in the following symbol:

function	symbol
20/13*[7]	ǂ˪♮↓

Table 2.55: Accidental symbol for function 20/13*[7]

What should appear at first glance when seeing the two symbols ǂ˪♮↓ et ǂ♮↓ is that the difference between the two comes down to the 7-identity accidental (˪), which indicates that the intonation between the two notes which would be thus altered must be adjusted by ear to represent the 7-identity.

2.5.2 Characteristic symbol

The following table shows families of tonalities with their characteristic symbols.

	1/	5/	7/
/1	1/1*	5/4*	7/4*
/5	8/5*		7/5*
/7	8/7*	10/7*	
/11	16/11*	20/11*	14/11*
/13	16/13*	20/13*	14/13*
/17	32/17*	20/17*	28/17*
/19	32/19*	20/19*	28/19*
/23	32/23*	40/23*	28/23*

Table 2.56: Characteristic symbols for each family of tonalities

To simplify the notation in general, it is preferable to use tonalities whose characteristic symbol refers to only one identity (in addition to the 3-identity). This is the case for families of tonalities which are found in the left column as well as on the upper row. For tonalities whose characteristic symbol refers to a combination of identities, we recommend that you limit yourself, as much as possible, to using identities 1-3 of the tonality in addition to its characteristic identity. For example, for tonalities 20/11* and 14/11*, these identities are 1-3-11.

2.5.3 Denomination

5-limit to 23-limit accidental symbols are simply referred to by their corresponding harmonic (abbreviated "h") or subharmonic (abbreviated "s").

For example, the symbol is called "harmonic 7," or "h7." For another example, the accidental is called "s13-h7-h5," or "s13-h35" if the harmonics 5 and 7 are multiplied together. Where appropriate, the name is followed by the specification "flat" or "sharp." If, in our example, the natural is replaced by a flat, the name becomes: "s13-h35-flat."

2.5.4 Summary tables

Once the contents of chapter 2 have been assimilated, it is possible to refer only to the two tables on the following page in order to be able to designate, note and play each note in the tonal network with the right intonation. For instruments of the brass family, it is more natural to tune in relation to *B*♭, rather than *A*: consequently, if a composition uses only brass instruments, it is preferable to use a tonal network having *B*♭ as the generating pitch (see chapter 5).

identity	symbol	cents	interval	3-limit note (see table opposite)						
23/	↑	+17	augmented fourth	B	F♯	C♯	G♯	D♯	A♯	E♯
19/	⸍	+3	minor third	A♭	E♭	B♭	F	C	G	D
17/	⸌	-9	augmented prime	F♯	C♯	G♯	D♯	A♯	E♯	B♯
13/	𝄳	-65	major sixth	D	A	E	B	F♯	C♯	G♯
11/	𝄲	+53	perfect fourth	B♭	F	C	G	D	A	E
7/	˪	-27	minor seventh	E♭	B♭	F	C	G	D	A
5/	♮↓	-22	major third	A	E	B	F♯	C♯	G♯	D♯
3/			perfect fifth	C	G	D	A	E	B	F♯
1	𝄫♭♮♯𝄪		↕	F	C	G	D	A	E	B
/3			perfect fifth	B♭	F	C	G	D	A	E
/5	♮↑	+22	major third	D♭	A♭	E♭	B♭	F	C	G
/7	˥	+27	minor seventh	G	D	A	E	B	F♯	C♯
/11	𝄳	-53	perfect fourth	C	G	D	A	E	B	F♯
/13	𝄲	+65	major sixth	A♭	E♭	B♭	F	C	G	D
/17	⸝	+9	augmented prime	F♭	C♭	G♭	D♭	A♭	E♭	B♭
/19	⸜	-3	minor third	D	A	E	B	F♯	C♯	G♯
/23	↓	-17	augmented fourth	C♭	G♭	D♭	A♭	E♭	B♭	F

Table 2.57: Summary table of accidental symbols and diatonic intervals

note	cents	note
A𝄪	+28	B♯
D𝄪	+26	E♯
G𝄪	+24	A♯
C𝄪	+22	D♯
F𝄪	+20	G♯
B♯	+18	C♯
E♯	+16	F♯
A♯	+14	B
D♯	+12	E
G♯	+10	A
C♯	+8	D
F♯	+6	G
B	+4	C
E	+2	F
A	-	B♭
D	-2	E♭
G	-4	A♭
C	-6	D♭
F	-8	G♭
B♭	-10	C♭
E♭	-12	F♭
A♭	-14	B𝄫
D♭	-16	E𝄫
G♭	-18	A𝄫
C♭	-20	D𝄫
F♭	-22	G𝄫
B𝄫	-24	
E𝄫	-26	
A𝄫	-28	

Table 2.58: 3-limit notes and corresponding cents deviations

3. Chords

Chapter 3 proposes a systematic method for naming chords, notating the dissonances incorporated therein and transcribing their sequences by modulation, paving the way for the understanding and effective use of the *Supplements* to the *Treatise*.

3.1 Preliminary concepts

3.1.1 Classification

Consider the following chord classification method:

— let us group together three identities, x, y and z, to form a chord [x : y : z];
— suppose that the order [x : y : z] is the arrangement which groups these identities together in the smallest possible interval, from the lowest to the highest note;
— let us perform all possible permutations of this triad.

The arrangements thus obtained are grouped into three pairs. Each pair bears the name of the identity at its basis as a capital letter: X, Y or Z. The first triad of each pair is in an interval lesser than an octave (tight arrangement); these triads can be designated by the index "1" (X_1, Y_1, Z_1). The second triad of each pair is in an interval greater than an octave (wide arrangement); these triads can be designated by the index "2" (X_2, Y_2, Z_2). These six permutations and their respective designations appear in the following table. Thus, the triad "X_1" corresponds to the arrangement [x : y : z]; "X_2" to the arrangement [x : z : y]; "Y_1" to the arrangement [y : z : x]; etc.

z	y	x	z	y	x
y	z	z	x	x	y
x	x	y	y	z	z
X_1	X_2	Y_1	Y_2	Z_1	Z_2

Table 3.1: Six arrangements of the triad [x : y : z]

Let us now apply this classification method to the following example:

— group together identities 1, 3 and 5 to form a chord;
— consider that the arrangement [4∶5∶6] is the one that groups these identities together in the smallest possible interval, from low to high;
— let us perform all possible permutations of this triad.

6	5	8	12	5	8
5	3	6	8	4	5
4	2	5	5	3	3
X_1	X_2	Y_1	Y_2	Z_1	Z_2

Table 3.2: Six arrangements of the triad [4∶5∶6]

Thus, triad "X_1" corresponds to arrangement [4∶5∶6]; "X_2" to arrangement [2∶3∶5]; "Y_1" to arrangement [5∶6∶8]; etc.

It is possible to add the fundamental at the bass to form a tetrad. Therefore, tetrad "X_1" corresponds to arrangement [2∶4∶5∶6]; "X_2" to arrangement [1∶2∶3∶5]; "Y_1" to arrangement [2∶5∶6∶8]; etc.

Generally speaking, it is preferable for the distance between the tenor and the bass of a chord to be between one or two octaves. This is why, in the example of tetrad [2∶5∶6∶8] above, the interval between bass and tenor is 2∶5 rather than 4∶5. However, if necessary, the gap between bass and tenor voices may be less than one octave or, exceptionally, increased to a 1∶6 interval. This flexibility of the bass makes it possible to adjust the arrangement of the chord according to the needs of sonority and instrumentation, and to modify the melodic motion of the bass to counterbalance that of the other voices (see section 3.2). The bass of a [x∶y∶z] triad can also be lowered by one or more octaves. For example, triad X_1 [4∶5∶6] can become [2∶5∶6] or, more exceptionally, [1∶5∶6].

3.1.2 Sequences of type X →

To optimize space in the *Supplements* to the *Treatise*, chord arrangements are presented in two rows (tight and wide arrangements) and three columns (X, Y, and Z arrangements). To facilitate tracking, the arrangements of the starting chords are always in the same place, while the positions of the arrangements of the destination chords vary according to the sequence presented. Here is the typical table of a sequence:

1 (tight arrangements)	X_1 → (X_1, or Y_1 or Z_1)	Y_1 → (X_1, or Y_1 or Z_1)	Z_1 → (X_1, or Y_1 or Z_1)
2 (wide arrangements)	X_2 → (X_2, or Y_2 or Z_2)	Y_2 → (X_2, or Y_2 or Z_2)	Z_2 → (X_2, or Y_2 or Z_2)
	X →	Y →	Z →

Table 3.3: Typical table for an X → chord sequence

It is not always possible to draw six arrangements from a sequence. This is especially true when chords become too dissonant for certain arrangements, which are rejected out of hand (see chapter 7). Even when only one arrangement is possible, the table retains the same three columns, but the unused row is removed.

3.1.3 Denomination

The name of any generic chord, regardless of its tonality or arrangement, corresponds to the enumeration, in ascending order, of its identities which are added to the fundamental. Thus, the chords [2:4:5:6], [1:2:3:5], [2:5:6:8], etc. are all called "3-5," since the presence of identities 3 and 5 characterizes their common sound.

In four specific cases, namely chords 3-5 and 3-5-15, as well as 3-7 and 3-7-9, the bass fundamental can be replaced, respectively, by identities 5 and 3 without causing too much dissonance. For example:

— [4:10:12:15] becomes [5:10:12:15];
— [4:5:6] becomes [5:10:12];
— [2:3:5] becomes [5:6:10];
— [4:7:12] becomes [6:7:12];
— [4:6:7:12] becomes [3:6:7:12];
— [4:6:7:9] becomes [3:6:7:9];
— etc.

On the perceptual level, by creating an octave and/or a fifth interval with another identity, the bass of these chords seems to have a fundamental function, which is not the case. At the same time, these arrangements highlight the sound of the interval 5:6 (316 c) of the chords 3-5 and 3-5-15, as well as the interval 6:7 (267 c) of the chords 3-7 et 3-7-9. These chords are therefore qualified as "minor," by analogy to the conventional diatonic minor third, whose width is similar (300 c). In this case, an "m" is added to the name of the chord: 3-5 m, 3-5-15 m, etc.

For the 3-5-15 m chord, the minor third sonority is represented by the 3-identity, the chord's only identity in black, regardless of the arrangement. As for the minor third sonority for the 3-7-9 m chord, it is represented by the 7-identity, the only identity of the chord in blue, no matter the arrangement.

For triads which only consist of two identities, we only use two arrangements (to prevent an octave from ending up in the two upper voices, causing a hollow tone): [2∶3∶4] or [1∶2∶3] for triad 3; [4∶5∶8] or [2∶4∶5] for triad 5; [5∶10∶12] or [5∶6∶10] for triad 3-5 m; etc.

3.1.4 Sequences of type W →

Triads composed of only two identities do not have permutations generating X, Y, Z arrangements, but only one W arrangement, tight or wide. The interval between the highest note and the lowest note of arrangement W_1 is equal to one octave, whereas this interval is greater than one octave for arrangement W_2. Arrangement W_1 is most often presented in a sequence that "opens" toward a wide arrangement, while arrangement W_2 is most often presented in a sequence that "closes" toward a tight arrangement, which allows generally to avoid too much parallelism between voices (see section 3.2). In a table, sequences in W arrangements are added as needed on horizontal lines, in no particular order. The prime symbol (') added to W means that this is the same starting chord arrangement segueing to a different destination chord arrangement. Here is the typical table of a sequence where a chord in arrangement W_1 is segued to a chord in arrangement W_2 (made of two identities) or to a chord in an X, Y or Z arrangement (made of three identities with various permutations):

1 (tight arrangements)	W_1 → (W_2, or X_2, or Y_2 or Z_2)	W'_1 → (W_2, or X_2, or Y_2 or Z_2)	W''_1 → (W_2, or X_2, or Y_2 or Z_2)
2 (wide arrangements)	W_2 → (W_1, or X_1, or Y_1 or Z_1)	W'_2 → (W_1, or X_1, or Y_1 or Z_1)	W''_2 → (W_1, or X_1, or Y_1 or Z_1)
	X →	Y →	Z →

Table 3.4: Typical table of a W → chord sequence

3.2 Melodic motion

This section will summarize traditional harmony-specific notions concerning melodic motion[1]. These notions aim to reinforce the cohesion of harmonic sequences as well as the identification and differentiation of voices between them. We will first observe three general considerations before describing three specific rules in more details.

3.2.1 General considerations

Here are three general considerations to look for:

— Presence of one or more common notes between the two chords;
— Melodic motions in conjunct intervals[2] or in intervals composed of consecutive harmonics such as, for example, 6∶7 (+267 c). The octave aside, no melodic leap should exceed a 5∶8 (814 c) minor sixth interval;
— Contrary motions between melodic voices. Above all, it is preferable that the bass be in motion contrary to all other voices; or, at the very least, to the highest voice of the chord.

Take for example the sequence [3∶4∶5] → [1∶2∶3], for which the tenor would be a note common to both chords.

To make it easier to visualize the melodic motions of each voice, let us transcribe the sequence vertically:

5 ↗ 3
4 = 2
3 ↘ 1

Knowing that the chord [1∶2∶3] is proportionally equivalent to [2∶4∶6],

5 ↗ 6
4 = 4
3 ↘ 2

[1] For a perspective on these rules through counterpoint, see Nicolas, 2013.

[2] A conjunct interval is equal to or lesser than 8∶9 (204 c). A larger interval is considered to be disjunct.

it is possible to make the following observations:

— the tenor is a note common to both chords (4 = 4, therefore 1∶1);
— the alto performs an ascending melodic motion (5 ↗ 6) between two consecutive harmonics (5∶6);
— the bass performs a descending melodic motion (3 ↘ 2) between two consecutive harmonics (3∶2).

In this example, the bass and alto voices proceed in contrary motions while each of these two voices proceeds in oblique motion in relation to the tenor.

3.2.2 Parallel motion

We forbid ourselves that two voices at a harmonic interval of an octave or perfect fifth in a starting chord retain this same interval in the destination chord. This motion is said to be “parallel” when both voices move in the same direction, and “consecutive” if they go in opposite directions.

In the example below, the interval between alto and bass in the starting chord is an octave (2∶4, or 1∶2); and, in the destination chord, that interval is still an octave (doubled, 2∶8, or 1∶4). This is a consecutive octave, a melodic motion that we forbid ourselves.

4 ↗ 8
3 ↘ 5
2 ↘ 2

3.2.3 Direct motion

We allow ourselves that two voices which proceed by a melodic motion in the same direction may arrive at a harmonic interval of an octave or a fifth provided that: the harmonic interval of these two voices in the starting chord is different, and; the upper voice of this interval proceeds by conjunct melodic motion. This motion is said to be “direct.”

In the example below, the interval between alto and bass in the starting chord is 2∶5; in the destination chord, the interval between these same voices is an octave (2∶4, or 1∶2). This last interval is created as both voices proceed by a melodic motion in the same direction. This is a direct octave. Assuming the alto proceeds by conjunct motion, this melodic motion is acceptable, though not ideal.

5 ↗ 4
4 ↘ 3
2 ↗ 2

In the *Supplements* to the *Treatise*, we present some tetrads performing a faulty or potentially problematic motion. In such cases, the voices involved are listed in grey. Omitting one of these voices can make a triad sequence possible.

This is the case in the example below, where it is possible to avoid a direct fifth between the soprano and the tenor of the destination tetrad.

5 ↘ 6
4 ↘ 5
3 ↘ 4
1 ↗ 2

Indeed, omitting one of the voices in grey will create one of the following two sequences of triads:

— without the soprano;

4 ↘ 5
3 ↘ 4
1 ↗ 2

— without the tenor.

5 ↘ 6
4 ↘ 5
1 ↗ 2

3.2.4 Unisons, crossings and overlaps

We avoid unisons and we refrain from using crossings and overlaps.

A unison (to be avoided) occurs when two voices are simultaneously on the same pitch. The two voices must arrive at this situation and leave it through contrary or oblique motions.

5 ↘ 9 = 9
4 ↗ ↘ 7
2 = 2 = 2

Crossings (forbidden) occur when, in a destination chord, a lower voice is found above a higher voice, or vice versa:

5 = 5
4 ↗ 6
2 = 2

Overlaps (forbidden) occur when, in a destination chord, a lower voice is found above the pitch (or on the same pitch) that the upper voice occupied in the starting chord, or vice versa:

5 ↗ 6
4 ↗ 5
2 = 2

Of course, all the rules that we have presented can be contradicted if the desired effect is, precisely, to blur the identification of voices. Moreover, the presence of a common note between two chords makes it possible to relax all the rules governing melodic motions, including those concerning the complexity of intervals in a modulation context.

3.3 Modulation

3.3.1 Attraction

Remember that tonality is the feeling of coherence emerging from a group of pitches in harmonic relationships (see section 1.1.3) and that the passage from one tonality to another is called modulation (see section 1.8). To justify leaving a starting tonality, a sufficient force of attraction must be exerted by the destination tonality. This force of attraction is exerted on the melodic and harmonic planes. Melodically, the more complex an interval, the smaller it should be[3]. Ratios with consecutive harmonics inherently follow this rule. For example: the interval 2 : 3 (+702 c) is very large, but also very simple on a tonal level, since it involves identities 1 and 3; the interval 77 : 78 (+22 c) is very small, but also tonally very complex, since it involves identities 3, 7, 11 and 13; etc.

However, these melodic ratios are not the only ones that can be used. For example, in some contexts the complex interval 32 : 35 (+155 c) may sound just as natural as the interval 2 : 3 (+702 c). This is particularly the case if it is hidden in an inner voice (the tenor or alto of

[3] This is explained by what we have already said about consonance (see section 1.1.4): the simpler the identity, the wider and stronger its tonal attraction field, so that any pitch can be considered as a relative dissonance if it falls within the tonal attraction field of a greater relative consonance.

a tetrad) and accompanied by other simpler melodic intervals (including a common note). Moreover, the simplicity of the modulation factor obviously influences the simplicity of the set of melodic intervals involved in a modulation.

Harmonically, the more the elements of a chord are organized in ascending order of identities, the greater the consonance of that chord. For example, the consonance of the chord 3-5-7 is greater in the arrangement [1:3:5:7] than in the arrangement [2:7:12:20][4]. Since a disorganized chord is more dissonant than an ordered one[5], the switch from the former to the latter creates a tension-resolving effect which seems to justify, *a posteriori*, the harmonic motion as a whole. Simply put, the need for novelty justifies the impoverishment of the arrangement. And the need to resolve the accumulated tension justifies the modulation.

3.3.2 Example A: one comparison for a single factor

Let us take the example of the sequence 3-5 → 3-5-7 by comparing the arrangements X → X (and, therefore, Y → Y and Z → Z) by a modulation factor of *5/3 (884 c), as shown in the table below. There are six arrangements possible for this sequence. For each arrangement, the melodic interval is indicated next to each voice. We did not colour code the melodic intervals in the tables, so as not to overly complicate their visualization. The modulation factor can be found at the top left of the table. Since this is a sequence with a common note between the two chords, the square to the left of the modulation factor is set in dark grey, rather than light grey (simple factor) or white (complex factor). To the right of the modulation factor, in small print, is a suggestion for an alternate melodic motion at the bass. In this example, if the bass identity of the starting chord is changed to 3, the melodic motion will be ascending towards the bass of the destination chord by an interval of 9:10 (+182 c), rather than 6:5 (-316 c), as shown in the table. To facilitate reading these alternatives, identities are always denoted by the smallest odd number possible, without considering octave ratios.

[4] The presence of high even numbers is enough to point out the identities' (litteral) disorder.

[5] This is for acoustic reasons, more precisely related to instrumental timbres. See Doty, 2002, pp. 9-24.

	*5/3 (884 c)				Bass: 3 ↗ 1	9:10 (+182)
1 (tight arrangements)	6 ↘ 7	36:35 (-49)	8 ↗ 10	24:25 (+71)	5 = 12	
	5 = 6		6 ↘ 7	36:35 (-49)	4 ↗ 10	24:25 (+71)
	4 ↗ 5	24:25 (+71)	5 = 6		3 ↘ 7	36:35 (-49)
	2 ↘ 2	6:5 (-316)	2 ↘ 2	6:5 (-316)	1 ↘ 2	6:5 (-316)
2 (wide arrangements)	5 = 12		12 ↘ 7	36:35 (-49)	8 ↗ 20	24:25 (+71)
	3 ↘ 7	36:35 (-49)	8 ↗ 5	24:25 (+71)	5 = 12	
	2 ↗ 5	24:25 (+71)	5 = 3		3 ↘ 7	36:35 (-49)
	1 ↘ 2	6:5 (-316)	2 ↘ 1	6:5 (-316)	2 ↘ 4	6:5 (-316)
	X →		Y →		Z →	

Table 3.5: X → X arrangements

Let us analyze the different sequence possibilities contained in this table:

— The Y_2 → sequence, or [2:5:8:12] → [1:3:5:7], probably provides the strongest sense of resolution, since the harmonics of the starting chord are out of order and those of the destination chord are perfectly ordered;

— The X_1 → sequence, that is to say [2:4:5:6] → [2:5:6:7], is balanced, since the harmonics of the two chords are almost perfectly ordered. The X_2 → sequence is of similar quality;

— The Y_1 → and Z_1 → sequences are of similar quality between them, but Y_1 → is preferable because of the 3-identity at the tenor voice of the destination chord, compared to the 7-identity for Z_1 →;

— The Z_2 → sequence is the least interesting one, because the two chords are disorganized, mostly the destination chord.

3.3.3 Example B : two comparisons for a single factor

In the *Supplements* to the *Treatise*, sequences are presented in tables put next to each other and arranged in ascending order of modulation factor. If, for the same modulation factor, the behaviour of the voices allows two chords to follow one another by more than one comparison of arrangements, a second table is added under the first without repeating the modulation factor. In the example below, the 3-5 chord is segued by comparing its own X → Z arrangements as well as X → Y by the *3/2 (702 c) modulation factor. The alternate bass motion is specified for the "(top)" and "(bottom)" sequences.

	*3/2 (702 c) Bass (top): 3 = 1 Bass (bottom): 1 ↗ 3 8:9 (+204)		
1 (tight arrangements)	6 ↗ 5 4:5 (+386) 5 ↗ 4 5:6 (+316) 4 ↗ 3 8:9 (+204) 2 ↘ 1 4:3 (-498)	8 ↗ 6 8:9 (+204) 6 ↗ 5 4:5 (+386) 5 ↗ 4 5:6 (+316) 4 ↘ 2 4:3 (-498)	5 ↗ 8 5:6 (+316) 4 ↗ 6 8:9 (+204) 3 ↗ 5 4:5 (+386) 2 ↘ 2 4:3 (-498)
2 (wide arrangements)	5 ↗ 8 5:6 (+316) 3 ↗ 5 4:5 (+386) 2 ↗ 3 8:9 (+204) 1 ↘ 1 4:3 (-498)	12 ↗ 5 4:5 (+386) 8 ↗ 3 8:9 (+204) 5 ↗ 2 5:6 (+316) 4 ↘ 1 4:3 (-498)	8 ↗ 12 8:9 (+204) 5 ↗ 8 5:6 (+316) 3 ↗ 5 4:5 (+386) 2 ↘ 2 4:3 (-498)
1 (tight arrangements)	6 = 8 5 ↘ 6 10:9 (-182) 4 ↘ 5 16:15 (-112) 2 ↗ 4 2:3 (+702)	8 ↘ 5 16:15 (-112) 6 = 4 5 ↘ 3 10:9 (-182) 2 ↗ 2 2:3 (+702)	5 ↘ 6 10:9 (-182) 4 ↘ 5 16:15 (-112) 3 = 4 1 ↗ 2 2:3 (+702)
2 (wide arrangements)	5 ↘ 12 10:9 (-182) 3 = 8 2 ↘ 5 16:15 (-112) 1 ↗ 4 2:3 (+702)	12 = 8 8 ↘ 5 16:15 (-112) 5 ↘ 3 10:9 (-182) 2 ↗ 2 2:3 (+702)	8 ↘ 5 16:15 (-112) 5 ↘ 3 10:9 (-182) 3 = 2 1 ↗ 1 2:3 (+702)
	X → Z X → Y	Y → X Y → Z	Z → Y Z → X

Table 3.6: X → X and X → Y arrangements

In the case of the W arrangement, a full vertical line is sometimes needed to signify that it is the same starting chord arrangement segued to a different destination chord arrangement. In the example below, the same chord [1:2:3] is segued to two different arrangements of the 3-7 chord by the *16/9 (996 c) modulation factor. Because this is a modulation without a common note, the square to the left of the modulation factor appears in light grey this time.

*16/9 (996 c)		
3 ↗ 7 27:28 (+63) 2 ↗ 6 3:4 (+498) 1 ↘ 2 9:8 (-204)	3 ↗ 7 27:28 (+63) 2 ↘ 4 9:8 (-204) 1 ↗ 3 3:4 (+498)	
X →	Y →	Z →

Table 3.7: W_2 → Z_1 and W_2 → X_2 arrangements

3.3.4 Example C: two comparisons for two factors

If two chords follow each other by more than one comparison of arrangements, but each of the comparisons uses different modulation factors, the sequence of tables is interrupted by a gap to mark where the change takes place. In the example below, the 3-5 chord is segued to the 3-11 chord by comparing the X → Z and X → Y arrangements using two different modulation factors. In this example, the square to the left of the *15/11 (537 c) modulation factor is left blank due to its complexity and the lack of a common note for these sequences.

	*15/11 (537 c)		
1 (tight arrangements)	6 ↗ 11 4:5 (+386) 5 ↗ 8 11:12 (+151) 4 ↗ 6 44:45 (+39) 2 ↘ 2 22:15 (-663)	8 ↗ 12 44:45 (+39) 6 ↗ 11 4:5 (+386) 5 ↗ 8 11:12 (+151) 4 ↘ 4 22:15 (-663)	5 ↗ 16 11:12 (+151) 4 ↗ 12 44:45 (+39) 3 ↗ 11 4:5 (+386) 2 ↘ 4 22:15 (-663)
2 (wide arrangements)	5 ↗ 16 11:12 (+151) 3 ↗ 11 4:5 (+386) 2 ↗ 6 44:45 (+39) 1 ↘ 4 22:15 (-663)	12 ↗ 11 4:5 (+386) 8 ↗ 6 44:45 (+39) 5 ↗ 4 11:12 (+151) 4 ↘ 2 22:15 (-663)	
(split) --	X → Z	Y → X	Z → Y

	*10/7 (617 c)		
1 (tight arrangements)	6 ↘ 16 21:20 (-84) 5 ↘ 12 7:6 (-267) 4 ↘ 11 56:55 (-31) 1 ↗ 4 7:10 (+617)	8 ↘ 11 56:55 (-31) 6 ↘ 8 21:20 (-84) 5 ↘ 6 7:6 (-267) 2 ↗ 4 7:10 (+617)	5 ↘ 12 7:6 (-267) 4 ↘ 11 56:55 (-31) 3 ↘ 8 21:20 (-84) 1 ↗ 4 7:10 (+617)
2 (wide arrangements)		12 ↘ 16 21:20 (-84) 8 ↘ 11 56:55 (-31) 5 ↘ 6 7:6 (-267) 2 ↗ 4 7:10 (+617)	8 ↘ 11 56:55 (-31) 5 ↘ 6 7:6 (-267) 3 ↘ 4 21:20 (-84) 1 ↗ 2 7:10 (+617)
	X → Y	Y → Z	Z → X

Table 3.8: X → Z and X → Y arrangements using two different modulation factors

3.4 Processing dissonances

3.4.1 Delay and anticipation

We know that non-prime odd number identities can always be interpreted as simpler identities in another tonality. For example, the 21-harmonic of 1-fundamental can be interpreted more simply as the 3-harmonic of 7-fundamental. The more complex the identity, the more its dissonance requires a particular melodic treatment to be tonally integrated in a coherent manner. In this section, we will detail the melodic treatments that we consider most appropriate for this purpose[6].

The 33- and 39-identities should be treated as delays. A delay takes the form of a consonance in the starting tonality before becoming, through modulation, a dissonance in the destination tonality. The dissonance thus created is resolved by melodic motion towards a consonance in the destination tonality.

The 21-, 27-, and 45-identities should be treated as delays or anticipations. An anticipation takes the form of a dissonance that appears by melodic motion in a starting chord. The dissonance is resolved, at the moment of a modulation, by taking the form of a consonance in the destination tonality.

The 9-, 15-, and 25-identities can be treated as delays or anticipations, or integrated to a stable chord[7], that is, a chord that does not require resolution of one of its identities on a stronger consonance.

[6] Here are the reasons why we accept or reject the integration of the most complex dissonances through a particular melodic treatment. Dissonances rejected on a melodic level can potentially be integrated harmonically as added notes (see section 3.4.6).

— The 33-identity (accepted) can be resolved simply at harmonic 32 (1-identity) by a *4/3 modulation factor;
— The 35-identity (rejected) requires a resolution on harmonic 36 (9-identity, not exclusive to the resolution tonality) by a *8/5 modulation factor, which we consider too complex to establish a sufficiently clear relationship;
— The 39-identity (accepted) is resolved, rather simply, on harmonic 40 (5-identity) by a *4/3 modulation factor;
— The 45-identity (accepted) resolves on harmonic 48 (3-identity) by a *16/9 modulation factor and by a melodic interval as simple as 15 : 16 (+112 c);
— The 49-identity (rejected), in addition to being complex in itself (7*7), requires a resolution on harmonic 48 (3-identity) by a *8/7 modulation factor and a very small melodic interval (36 c). Such a small interval between a dissonance and its resolution note may make the dissonance appear as a mere mistake in intonation;
— Identities higher than 49 do not provide more favorable conditions.

[7] The inclusion of the 25-identity in this group may be surprising, given its relative dissonance. However, 25 is a 5-limit identity that is not too close to a strong consonance, unlike 21 (729 c) compared to 3 (702 c), for example. Furthermore, analysing 25 as part of a stable (even if dissonant) chord greatly facilitates its use and its identification in the *Supplements*.

Delays and anticipations may be described in the form of an equation. The number on the left side of the equation refers to the starting chord and the number on the right side to the destination chord. The mnemonic logic for the notation of delays and anticipations is as follows: a parenthesis surrounding the first factor of the equation signals an anticipation; a parenthesis surrounding the last factor of the equation signals a delay.

3.4.2 An example of delay

Here is an example of a sequence with a possible delay (on the left) and its realization (on the right).

			*16/9 (996c)
8	↗	5	9:10 (+182)
6	↗	4	27:32 (+294)
5	↗	3	15:16 (+112)
2	↘	1	9:8 (-204)

	*16/9			
8	=	9	↗	5
6	↗	8	=	4
5	↗	6	=	3
2	↘	2	=	1

Table 3.9: Sequence with a possible delay and its realization

This delay may be described by the equation 1 = (9), which tells us that the 1-identity of the starting chord, which is at the soprano, becomes, by modulation, the 9-identity of the destination chord. The 9-identity is placed in parentheses because it has a delay function. The delay is created by a modulation (indicated at the top of the vertical dotted line) and the dissonance is resolved on the 5-harmonic of the destination chord by the melodic interval 9:10 (+182 c). Above, the 9-harmonic is highlighted to help you see the delay.

3.4.3 An example of anticipation

If we now reverse the sequence, we are faced with a possible anticipation (9) = 1.

			*9/8 (204c)
5	↘	8	10:9 (-182)
4	↘	6	32:27 (-294)
3	↘	5	16:15 (-112)
1	↗	2	8:9 (+204)

			*9/8		
5	↘	9	=	8	
4		8	↘	6	32:27 (-294)
3		6	↘	5	16:15 (-112)
1		2	↗	2	8:9 (+204)

Table 3.10: Example of a sequence with a possible anticipation and its realization

The equation (9) = 1 tells us that the 9-identity of the starting chord becomes the 1-identity in the destination chord. The 9-identity is placed in parentheses because it has an anticipation function. This anticipation is created by the melodic interval 10 : 9 (-182 c), and the dissonance resolves through modulation (indicated above by the vertical dotted line) by becoming the 8-harmonic of the destination chord. Above, the 9-harmonic is highlighted to help you see the anticipation.

3.4.4 Resolution

A resolution is thus always carried out on the identity which is as simple and as close as possible to the dissonant pitch; a descending melodic motion is preferable, although an ascending one is also possible. In the preceding delay example, 1 = (9), the resolution by an ascending motion on the 5-identity is required to avoid a consecutive octave with the bass that would occur with a descending motion to the 4-harmonic.

What differentiates anticipations and delays from all other dissonances is that the dissonant note can be integrated into a transitional chord, that is to say a chord in which a dissonance lasts a certain time without its resolution note being present. As such, whether it occurs by melodic motion or by modulation, the resolution of a non-prime odd number identity can be avoided, during modulation, by assuming the role of a note common to the starting and destination chords.

In the following example, the delay is resolved through modulation without any melodic motion of the soprano voice, which becomes a consonance (6-harmonic) in the final chord.

	*16/9	⁞		*3/2	⁞		
8	=	⁞	9	=	⁞	6	
6	↗	⁞	8	↘	⁞	5	16 : 15 (-112)
5	↗	⁞	6	=	⁞	4	
2	↘	⁞	2	↗	⁞	2	2 : 3 (+702)

Table 3.11: Example of a delay resolved through modulation

3.4.5 Use of parentheses

In transcribing a sequence, we suggest putting a note in parentheses to disregard its octave, which is then simply deduced from its position among the voices. For example, the pitch “(x)” is found: at the soprano of the chord [1 : 2 : 3 : (x)]; at the alto of the chord [1 : 2 : (x) : 3]; etc. This preserves the notation of the structural chord, apart from its passing pitch.

For example, this resolution of a delay by the descending melodic interval 21:20 (-84 c)

24	=	6	
21	↘	5	21:20 (-84)
8	=	2	

is easier to visualize like this.

6	=	6	
(21)	↘	5	21:20 (-84)
2	=	2	

3.4.6 Added note

Parentheses are particularly useful for added notes. A note is said to be "added" when it creates dissonance by coexisting with its resolution note, which is different from a delay or an anticipation. Depending on whether the added note is in a harmonic or subharmonic position with respect to the fundamental, its notation will either take the form of a denominative (subharmonic) or of an odd number as small as possible (harmonic).

For example, an added note at a distance of a 63:64 (27 c) interval above the 7-identity of a chord corresponds to subharmonic 16/9 of the fundamental. The chord is noted as follows: [2:7:(16/9):8]. Conversely, a note added at a distance of a 64:63 (27 c) interval below the 8-identity of a chord corresponds to harmonic 63 of the fundamental. This time, the chord is noted as follows: [2:(63):8:9].

Just like what we saw for delays and anticipations, additions may be described by an equation. In the example below: the 8-harmonic in the starting tonality becomes the 16/9-subharmonic in the destination tonality, which would be indicated by 1 = (16/9); the 63-harmonic of the starting tonality becomes the 7-harmonic in the destination tonality, which would be indicated by (63) = 7.

	*9/8 (204 c)
9 = 8 8 ↘ 7 64:63 (-27) 2 ↗ 2 8:9 (+204)	

	*9/8		
9	=	8	
8	=	(16/9)	
(63)	=	7	
2	↗	2	8:9 (+204)

Table 3.12: Sequence with possible added notes and its realization

3.4.7 Pivot chord

Finally, the analysis of a sequence determines the location of the modulation and modifies the notation of anticipations and delays accordingly. For example, let us take the sequence 3-7 m → 3 by a modulation factor of *7/4 (969 c).

	*7/4 (969 c)
7 = 4	
6 ↘ 3	8 : 7 (-231)
3 ↗ 2	6 : 7 (+267)

Table 3.13: 3-7 m → 3 sequence by a *7/4 (969 c) modulation factor

The same sequence can be analyzed in two ways involving different voices:

— with an anticipation at the bass;

8/7*	*7/4	1/1*
7 =	7 =	4
6 =	6 ↘	3
3 ↗	(7) =	2

Table 3.14: Analysis of a modulation with an anticipation at the bass

— with a delay at the tenor.

8/7*	*7/4	1/1*	
7 =		4	= 4
6 =		(12/7)	↘ 3
3 ↗		2	= 2

Table 3.15: Analysis of a modulation with a delay at the tenor

In both cases, the chords [(7) : 6 : 7] or [2 : (12/7) : 4] are considered to be the “pivot chords” of the modulation, precisely because they can be interpreted in two different tonalities.

3.5 Transcribing sequences

As we explained in section 3.1.1, harmonic sequences can be transcribed by arranging chords either vertically or horizontally. The vertical layout is more suitable for analytical transcription, while the horizontal one is better for schematic transcriptions.

3.5.1 Analytical transcription

For example, let us take the sequence 3-7 → 3-5 by a modulation factor of *4/3 (498 c).

			*4/3 (498 c)
7	↘	5	21 : 20 (-84)
4	=	3	
3	↘	2	9 : 8 (-204)
1	↗	1	3 : 4 (+498)

Table 3.16: 3-7 → 3-5 by a *4/3 (498 c) modulation factor

Let us imagine that the first chord, [1 : 3 : 4 : 7], is gradually built, one voice at a time, then that the modulation is carried out by successive melodic motions generating the anticipations and delays that we want to transcribe. Below, from left to right, the columns always represent successive stages.

						*4/3			
			7		7	=	(21)	↘	5
1	2	4	4		4	=	3		3
		3	3	↘	(4/3)	=	2		2
	1	1	1		1	↗	1		1

Table 3.17: Example of an analytical transcription

We note that the addition of the bass and tenor voices successively reinterprets the octave of the alto as 1-, 2-, and 4-harmonics. We could have noted the alto as a 4-harmonic from the start. The first melodic motion of the sequence, which is at the tenor, creates an anticipation of the destination tonality in the starting tonality. Then, the melodic motion of the bass provokes simultaneously the feeling: of modulation; of resolving the anticipation of the tenor on the 2-harmonic in the destination tonality; of a delay at the soprano on the 21-harmonic in the destination tonality.

The space above the melodic motions can be used to specify other information than just the modulation factor, namely: specific tonalities, a summary of the harmonic sequence, and the names of the chords. So, from top to bottom, we find increasingly detailed levels of analysis.

				3-7 → 3-5						
3/2*			[1:3:4:7]			*4/3	⁝	1/1*		[1:3:4:5]
			[7]		7	=	⁝	(21)	↘	[5]
[1]	2	4	4		4	=	⁝	3		3
		[3]	3	↘	(4/3)	=	⁝	2		2
	[1]	1	1		1	↗	⁝	[1]		1

Table 3.18: Example of an analytical transcription with different levels of analysis

Indications above the horizontal line can be transcribed in a score above the staff. As for the identities found under the horizontal line, they can be transcribed for each note in the corresponding staff. The bracket framing ("[ ]") of the identity of each new note makes it easier to visualize the harmonic progression in this context.

3.5.2 Schematic transcription

Transcribing chords horizontally provides more continuous space. This is how the previous sequence would look. Below, from top to bottom, the lines represent successive stages.

	...	
3/2*	4	
	1:4	
	1:3:4	
	1:3:4:7	
	1:3:(4/3):7	*4/3 (498 c)
1/1*	1:3:4:(21)	
	1:3:4:5	
	...	

Table 3.19: Example of a schematic transcription

Transcribing chords horizontally comes at the expense of a clear visualization of voice motions, which can be a benefit when simplifying the sketching of a passage. If necessary, lines can be added by hand to indicate voice continuities. This diagram can be simplified further by including only certain information, such as:

— the harmonic summary;

3/2*	… 1 : 3 : 4 : 7	*4/3 (498 c)
1/1*	1 : 3 : 4 : 5 …	

Table 3.20: Example of a schematic transcription using the harmonic summary

— the names of the chords.

3/2*	… 3 3-7	*4/3 (498 c)
1/1*	3-5 …	

Table 3.21: Example of a schematic transcription using chord names

To schematize the transcription even further, the harmonic content of entire sections of a work can be summed up by the identities involved. For example, the previous sequence could be part of a "3-5-7" sequence and be distinguished from the following sections, "3-5-25" and "3-19."

1

Part Two

4. Tonal network (strings)

All the notes of the tonal network as defined previously (see section 1.6) are presented in the tables found in this chapter. Each table presents a family of tonalities and bears the name of that family. From bottom to top, lines represent identities 1 through 23 (excluding multiples of 3) in ascending order. From left to right, columns show multiples of 3 in ascending order.

In the left margin, some identities are accompanied by indications about tonality and identity. For example, the 7-identity of the 8/5* table is accompanied by "7/5*[1]." This indicates that the notes which represent the 7-identities of the 8/5* table are the same that represent the 1-identity of the 7/5* table. 25-identities are indicated in the same fashion. Also, the identities of non-prime numbers that are multiples of 3 (15, 21, 27, 33, 39, and 45) can all be read directly in the same table.

The notes of the network are limited as a function of the reference notes, that is to say notes whose pitch is theoretically given accurately without the need for performers to adjust their playing. Take, for example, the open strings of the cello.

string	factors	denominative	note
I	1/1	1/1	*A* +0
II	1/3	4/3	*D* -2
III	$1/3^2$	16/9	*G* -4
IV	$1/3^3$	32/27	*C* -6

Table 4.1: Values of the open strings of a cello

Each string produces reference notes whose identities are multiples of 1 (vibration in fundamental mode), 3, 5, and 7 (vibration in harmonic mode). In a table, if one lays out the multiples of 3 horizontally, all that remains is to represent harmonics 1, 5, and 7 vertically. Below, the result of this classification is expressed as factors.

	string IV	string III	string II	string I	
7-harmonic	$7/3^3$	$7/3^2$	7/3	7/1	
5-harmonic	$5/3^3$	$5/3^2$	5/3	5/1	
1-harmonic	$1/3^3$	$1/3^2$	1/3	1/1	3/1

Table 4.2: Cello reference notes expressed as factors

Below, the same results are expressed as denominatives.

	string IV	string III	string II	string I	
7-harmonic	28/27	14/9	7/6	7/4	
5-harmonic	40/27	10/9	5/3	5/4	
1-harmonic	32/27	16/9	4/3	1/1	3/2

Table 4.3: Cello reference notes expressed as denominatives

This gives us a total of thirteen reference notes. In the tables of chapters 4 and 5, these notes are always highlighted in yellow to make them easier to find.

7	28/27 L♭B 63 -37	14/9 LF 765 -35	7/6 LC 267 -33	7/4 LG 969 -31	
5	40/27 ♮↓E 680 -20	10/9 ♮↓B 182 -18	5/3 ♯↓F 884 -16	5/4 ♯↓C 386 -14	
1	32/27 ♮C 294 -6	16/9 ♮G 996 -4	4/3 ♮D 498 -2	1/1 ♮A 0 +0	3/2 ♮E 702 +2

Table 4.4: Values of the thirteen reference notes for a tuning fork at *A* = 1/1

The range of the tonal network is limited according to the tonal relationships allowed by the reference notes. Indeed, all the notes of the network must be able to integrate a reference note in a chord without exceeding the prime number 23 or the non-prime odd number 21.

At the same time, so as not to overly complicate the tones of the chords, the network is also limited according to the following arbitrary considerations and restrictions: tonalities whose characteristic identity is 7 do not contain identities greater than 17; tonalities whose characteristic identity is 11, 13, or 17 do not contain other prime numbers identities greater than 7; tonalities whose characteristic identity is 19 or 23 do not contain other prime numbers identities greater than 5.

Finally, some cells are displayed in grey for two reasons: their tonality implies the odd non-prime numbers 9, 15 or 21[1], or an additional reference note to those mainly retained[2], or; they require too many accidentals (see section 2.5.2).

[1] In the summary tables for the string and brass instruments tonal networks (see sections 4.1 and 5.1), these tonalities are also displayed in grey.

[2] For example: for strings, the *E* string of the double bass and its harmonics; for brass instruments, the *C* trumpet and its harmonics; etc.

4.1 Summary table

	♮ /1	♮ 5/	7/
/1 ♮	9/8* 3/2* 1/1* 4/3* 16/9* 32/27* 128/81* 256/243*	- 15/8* 5/4* 5/3* 10/9* 40/27* 160/81* 320/243*	- 21/16* 7/4* 7/6* 14/9* 28/27* 112/81* 448/243*
/5 ♮	9/5* 6/5* 8/5* 16/15* 64/45* 256/135*		- 21/20* 7/5* 28/15* 56/45* 224/135*
/7	9/7* 12/7* 8/7* 32/21* 64/63* 256/189*	- 15/14* 10/7* 40/21* 80/63* 320/189*	
/11	12/11* 16/11* 64/33* 128/99* 512/297*	- 20/11* 40/33* 160/99* 320/297*	- 14/11* 56/33* 112/99* 448/297*
/13	24/13* 16/13* 64/39* 128/117* 512/351*	- 20/13* 40/39* 160/117* 640/351*	- 14/13* 56/39* 224/117* 448/351*
/17	24/17* 32/17* 64/51* 256/153* 512/459*	- 20/17* 80/51* 160/153* 640/459*	- 28/17* 56/51* 224/153* 896/459*
/19	24/19* 32/19* 64/57* 256/171* 1024/513*	- 20/19* 80/57* 320/171* 640/513*	- 28/19* 112/57* 224/171* 896/513*
/23	24/23* 32/23* 128/69* 512/267* 1024/621*	- 40/23* 80/69* 320/207* 640/621*	- 28/23* 112/69* 448/267* 896/621*

4.2 Order 1

4.2.1 1/1* Family

23 ↑		92/81 ↑B 220 +20	46/27 ↑♯F 922 +22	23/18 ↑♯C 424 +24	23/12 ↑♯G 1126 +26	23/16 ↑♯D 628 +28	69/64 ↑♯A 130 +30	207/128 ↑♯E 832 F+32		
19 ⸝	304/243 ⸝♭D 388 -12	152/81 ⸝♭A 1090 -10	38/27 ⸝♭E 592 -8	19/18 ⸝♭B 94 -6	19/12 ⸝F 796 -4	19/16 ⸝C 298 -2	57/32 ⸝G 999 -1	171/128 ⸝D 501 +1		
17 ⸜⸜	272/243 ⸜⸜B 195 -5	136/81 ⸜⸜♯F 897 -3	34/27 ⸜⸜♯C 399 -1	17/9 ⸜⸜♯G 1101 +1	17/12 ⸜⸜♯D 603 +3	17/16 ⸜⸜♯A 105 +5	51/32 ⸜⸜♯E 807 F+7	153/128 ⸜⸜♯B 309 C+9		
13 𝄳	416/243 𝄳G 931 G♭+31	104/81 𝄳D 433 D♭+33	52/27 𝄳A 1135 A♭+35	13/9 𝄳E 637 E♭+37	13/12 𝄳B 139 B♭+39	13/8 𝄳♯F 841 F+41	39/32 𝄳♯C 342 C+42	117/64 𝄳♯G 1044 G+44		
11 𝄲	352/243 𝄲♭E 642 +42	88/81 𝄲♭B 143 +43	44/27 𝄲F 845 +45	11/9 𝄲C 347 +47	11/6 𝄲G 1049 +49	11/8 𝄲D 551 D♯-49	33/32 𝄲A 53 A♯+47	99/64 𝄲E 755 F-45		
7 ⌊	448/243 ⌊♭A 1059 -41	112/81 ⌊♭E 561 -39	28/27 ⌊♭B 63 -37	14/9 ⌊F 765 -35	7/6 ⌊C 267 -33	7/4 ⌊G 969 -31	21/16 ⌊D 471 -29	63/32 ⌊A 1173 -27		
5 ♮↓ 8/5*[25]	320/243 ♮↓D 477 -23	160/81 ♮↓A 1178 -22	40/27 ♮↓E 680 -20	10/9 ♮↓B 182 -18	5/3 ♯↓F 884 -16	5/4 ♯↓C 386 -14	15/8 ♯↓G 1088 -12	45/32 ♯↓D 590 -10		
1 ♮	256/243 ♭B 90 -10	128/81 ♮F 792 -8	32/27 ♮C 294 -6	16/9 ♮G 996 -4	4/3 ♮D 498 -2	1/1 ♮A 0 +0	3/2 ♮E 702 +2	9/8 ♮B 204 +4	27/16 ♯F 906 +6	81/64 ♯C 408 +8

4.2.2 8/5* Family

23		184/135 ♮D 536 +36	46/45 ♮A 38 +38	23/15 ♮E 740 +40	23/20 ♮B 242 +42	69/40 ♯F 944 +44	207/160 ♯C 446 +46		
19	608/405 ♭F 703 E+3	152/135 ♭C 205 B+5	76/45 ♭G 907 +7	19/15 ♭D 409 +9	19/10 ♭A 1111 +11	57/40 ♭E 613 +13	171/160 ♭B 115 +15		
17		136/135 ♮A 13 +13	68/45 ♮E 715 +15	17/15 ♮B 217 +17	17/10 ♯F 919 +19	51/40 ♯C 421 +21	153/80 ♯G 1123 +23		
13		208/135 ♮F 748 E+48	52/45 ♮C 250 -50	26/15 ♮G 952 -48	13/10 ♮D 454 -46	39/20 ♮A 1156 -44	117/80 ♮E 658 -42		
11		176/135 ♭D 459 D-41	88/45 ♭A 1161 A-39	22/15 ♭E 663 E-37	11/10 ♭B 165 B-35	33/20 ♮F 867 F♯-33	98/80 ♮C 369 C♯-31		
7 7/5*[1]	448/405 ♭C 175 B-25	224/135 ♭G 877 -23	56/45 ♭D 379 -21	28/15 ♭A 1081 -19	7/5 ♭E 583 -17	21/20 ♭B 84 -16	63/40 ♮F 786 -14		
5 ♮	128/81 ♮F 792 -8	32/27 ♮C 294 -6	16/9 ♮G 996 -4	4/3 ♮D 498 -2	1/1 ♮A 0 +0	3/2 ♮E 702 +2	9/8 ♮B 204 +4		
1 ♮	512/405 ♭D 406 +6	256/135 ♭A 1108 +8	64/45 ♭E 610 +10	16/15 ♭B 112 +12	8/5 ♮F 814 +14	6/5 ♮C 316 +16	9/5 ♮G 1018 +18	27/20 ♮D 520 +20	81/80 ♮A 22 +22

4.2.3 8/7* Family

17	272/189 ♯D 630 +30	68/63 ♯A 132 +32	34/21 ♯E 834 F+34	17/14 ♯B 336 C+36	51/28 ×F 1038 G+38	153/112 ×C 540 D+40		
13	208/189 B 166 -34	104/63 ♯F 868 -32	26/21 ♯C 370 -30	13/7 ♯G 1072 -28	39/28 ♯D 574 -26	117/112 ♯A 76 -24		
11	352/189 G 1077 G♯-23	88/63 D 579 D♯-21	22/21 A 81 A♯-19	11/7 E 782 F-18	33/28 B 284 C-16	99/56 ♯F 986 G-14		
7	32/27 ♮C 294 -6	16/9 ♮G 996 -4	4/3 ♮D 498 -2	1/1 ♮A 0 +0	3/2 ♮E 702 +2	9/8 ♮B 204 +4		
5 ♮ 10/7*[1]	320/189 ♯F 912 +12	80/63 ♯C 414 +14	40/21 ♯G 1116 +16	10/7 ♯D 617 +17	15/14 ♯A 119 +19	45/28 ♯E 821 F+21		
1 ♮	256/189 D 525 +25	64/63 A 27 +27	32/21 E 729 +29	8/7 B 231 +31	12/7 ♯F 933 +33	9/7 ♯C 435 +35	27/14 ♯G 1137 +37	81/56 ♯D 639 +39

4.2.4 16/11* Family

11	32/27 ♮C 294 -6	16/9 ♮G 996 -4	4/3 ♮D 498 -2	1/1 ♮A 0 +0	3/2 ♮E 702 +2	9/8 ♮B 204 +4	
7 ʟ 14/11*[1]	448/297 𝄳ʟF 712 E+12	112/99 𝄳ʟC 214 B+14	56/33 𝄳ʟG 916 G♭+16	14/11 𝄳ʟD 418 D♭+18	21/11 𝄳ʟA 1119 A♭+19	63/44 𝄳ʟE 621 E♭+21	
5 ♮↓ 20/11*[1]	320/297 𝄳♮↓B 129 B♭+29	160/99 𝄳♯↓F 831 F+31	40/33 𝄳♯↓C 333 C+33	20/11 𝄳♯↓G 1035 G+35	15/11 𝄳♯↓D 537 D+37	45/44 𝄳♯↓A 39 A+39	135/88 𝄳♯↓E 741 E+41
1 ♮	512/297 𝄳G 943 G♭+43	128/99 𝄳D 445 D♭+45	64/33 𝄳A 1147 A♭+47	16/11 𝄳E 649 E♭+49	12/11 𝄳B 151 -49	18/11 𝄳♯F 853 -47	27/22 𝄳♯C 355 -45

4.2.5 16/13* Family

13	32/27 ♮C 294 -6	16/9 ♮G 996 -4	4/3 ♮D 498 -2	1/1 ♮A 0 +0	3/2 ♮E 702 +2	9/8 ♮B 204 +4	
7 ʟ 14/13*[1]	448/351 ♯ʟ♭D 422 +22	224/117 ♯ʟ♭A 1124 +24	56/39 ♯ʟ♭E 626 +26	14/13 ♯ʟ♭B 128 +28	21/13 ♯ʟF 830 +30	63/52 ♯ʟC 332 +32	
5 ♮↓ 20/13*[1]	640/351 ♯♮↓G 1040 +40	160/117 ♯♮↓D 542 +42	40/39 ♯♮↓A 44 +44	20/13 ♯♮↓E 746 +46	15/13 ♯♮↓B 248 +48	45/26 ♯♯↓F 950 +50	135/104 ♯♯↓C 452 D-48
1 ♮	512/351 ♯♭E 654 E-46	128/117 ♯♭B 156 B-44	64/39 ♯F 858 F♯-42	16/13 ♯C 359 C♯-41	24/13 ♯G 1061 G♯-39	18/13 ♯D 563 D♯-37	27/26 ♯A 65 A♯-35

4.2.6 32/17* Family

17	32/27 ♮C 294 -6	16/9 ♮G 996 -4	4/3 ♮D 498 -2	1/1 ♮A 0 +0	3/2 ♮E 702 +2	9/8 ♮B 204 +4	
7 28/17*[1]	896/459 ♭♭B 1158 A-42	224/153 ♭F 660 E-40	56/51 ♭C 162 B-38	28/17 ♭G 864 -36	21/17 ♭D 366 -34	63/34 ♭A 1068 -32	
5 ♮ 20/17*[1]	640/459 ♭E 575 -25	160/153 ♭B 77 -23	80/51 ♮F 779 -21	20/17 ♮C 281 -19	30/17 ♮G 983 -17	45/34 ♮D 485 -15	
1 ♮	512/459 ♭C 189 B-11	256/153 ♭G 891 -9	64/51 ♭D 393 -7	32/17 ♭A 1095 -5	24/17 ♭E 597 -3	18/17 ♭B 99 -1	27/17 F 801 +1

4.2.7 32/19* Family

19	32/27 ♮C 294 -6	16/9 ♮G 996 -4	4/3 ♮D 498 -2	1/1 ♮A 0 +0	3/2 ♮E 702 +2	9/8 ♮B 204 +4		
5 ♮ 20/19*[1]	640/513 ♯C 383 -17	320/171 ♯G 1085 -15	80/57 ♯D 587 -13	20/19 ♯A 89 -11	30/19 ♯E 791 F-9	45/38 ♯B 293 C-7	135/76 𝄪F 995 G-5	
1 ♮	1024/513 A 1197 -3	256/171 E 699 -1	64/57 B 201 +1	32/19 ♯F 902 +2	24/19 ♯C 404 +4	36/19 ♯G 1106 +6	27/19 ♯D 608 +8	81/76 ♯A 110 +10

4.2.8 32/23* Family

23 ↑	32/27 ♮C 294 -6	16/9 ♮G 996 -4	4/3 ♮D 498 -2	1/1 ♮A 0 +0	3/2 ♮E 702 +2	9/8 ♮B 204 +4	
5 ♮↓ 40/23*[1]	640/621 ↓↓♭B 52 -48	320/207 ↓↓♮F 754 -46	80/69 ↓↓♮C 256 -44	40/23 ↓↓♮G 958 -42	30/23 ↓↓♮D 460 -40	45/23 ↓↓♮A 1162 -38	
1 ♮	1024/621 ↓♭G 866 -34	256/207 ↓♭D 368 -32	128/69 ↓♭A 1070 -30	32/23 ↓♭E 572 -28	24/23 ↓♭B 74 -26	36/23 ↓F 776 -24	27/23 ↓C 278 -22

4.3 Order 5

4.3.1 5/4* Family

23		115/81 ♯D 607 +7	115/108 ♯A 109 +9	115/72 ♯E 811 F+11	115/96 ♯B 313 C+13	115/64 𝄪F 1015 G+15			
19		95/81 ♮C 276 -24	95/54 ♮G 978 -22	95/72 ♮D 480 -20	95/48 ♮A 1182 -18	95/64 ♮E 684 -16	285/256 ♮B 186 -14		
17		85/81 ♯A 83 -17	85/54 ♯E 785 F-15	85/72 ♯B 287 C-13	85/48 𝄪F 989 G-11	85/64 𝄪C 491 D-9			
13	260/243 ♮B 117 B♭+17	130/81 ♯F 819 F+19	65/54 ♯C 321 C+21	65/36 ♯G 1023 G+23	65/48 ♯D 525 D+25	65/64 ♯A 27 A+27			
11	440/243 ♮G 1028 +28	110/81 ♮D 530 +30	55/54 ♮A 32 +32	55/36 ♮E 734 +34	55/48 ♮B 236 +36	55/32 ♯F 938 +38			
7 7/4*[5]	280/243 ♮C 245 B+45	140/81 ♮G 947 G♭+47	35/27 ♮D 449 D♭+49	35/18 ♮A 1151 -49	35/24 ♮E 653 -47	35/32 ♮B 155 -45	105/64 ♯F 857 -43		
5 ♮ 1/1*[25]	400/243 ♯F 863 -37	100/81 ♯C 365 -35	50/27 ♯G 1067 -33	25/18 ♯D 569 -31	25/24 ♯A 71 -29	25/16 ♯E 773 F-27	75/64 ♯B 275 C-25		
1 ♮ 8/5*[25]	320/243 ♮D 477 -23	160/81 ♮A 1178 -22	40/27 ♮E 680 -20	10/9 ♮B 182 -18	5/3 ♯F 884 -16	5/4 ♯C 386 -14	15/8 ♯G 1088 -12	45/32 ♯D 590 -10	135/128 ♯A 92 -8

4.3.2 10/7* Family

7	40/27 ♮E 680 -20	10/9 ♮B 182 -18	5/3 ♯F 884 -16	5/4 ♯C 386 -14	15/8 ♯G 1088 -12		
5 ♮ 8/7*[25]	200/189 ♯A 98 -2	100/63 ♯E 800 F+0	25/21 ♯B 302 C+2	25/14 𝄪F 1004 G+4	75/56 𝄪C 506 D+6		
1 ♮ 8/7*[5]	320/189 ♯F 912 +12	80/63 ♯C 414 +14	40/21 ♯G 1116 +16	10/7 ♯D 617 +17	15/14 ♯A 119 +19	45/28 ♯E 821 F+21	135/112 ♯B 323 C+23

4.3.3 20/11* Family

11	40/27 ♮E 680 -20	10/9 ♮B 182 -18	5/3 ♯F 884 -16	5/4 ♯C 386 -14	15/8 ♯G 1088 -12		
7 14/11*[5]	560/297 ♮A 1098 A♭-2	140/99 ♮E 600 E♭+0	35/33 ♮B 102 B♭+2	35/22 ♯F 804 F+4	105/88 ♯C 306 C+6		
5 ♮ 16/11*[25]	400/297 ♯D 515 D+15	100/99 ♯A 17 A+17	50/33 ♯E 719 E+19	25/22 ♯B 221 B+21	75/44 𝄪F 923 F♯+23		
1 ♮ 16/11*[5]	320/297 ♮B 129 B♭+29	160/99 ♯F 831 F+31	40/33 ♯C 333 C+33	20/11 ♯G 1035 G+35	15/11 ♯D 537 D+37	45/44 ♯A 39 A+39	135/88 ♯E 741 E+41

4.3.4 20/13* Family

13	40/27 ♮E 680 -20	10/9 ♮B 182 -18	5/3 ♯F 884 -16	5/4 ♯C 386 -14	15/8 ♯G 1088 -12		
7 14/13*[5]	560/351 ♯♮F 809 +9	140/117 ♯♮C 311 +11	70/39 ♯♮G 1013 +13	35/26 ♯♮D 515 +15	105/104 ♯♮A 17 +17		
5 16/13*[25]	400/351 ♯♮D 226 +26	200/117 ♯♮A 928 +28	50/39 ♯♮E 430 +30	25/13 ♯♮B 1132 +32	75/52 ♯♯F 634 +34		
1 16/13*[5]	640/351 ♯♮G 1040 +40	160/117 ♯♮D 542 +42	40/39 ♯♮A 44 +44	20/13 ♯♮E 746 +46	15/13 ♯♮B 248 +48	45/26 ♯♯F 950 +50	135/104 ♯♯C 452 D-48

4.3.5 20/17* Family

17	40/27 ♮E 680 -20	10/9 ♮B 182 -18	5/3 ♯F 884 -16	5/4 ♯C 386 -14	15/8 ♯G 1088 -12	
7 28/17*[5]	560/459 ♭D 344 C+44	280/153 ♭A 1046 G+46	70/51 ♭E 548 D+48	35/34 ♭B 50 -50	105/68 ♮F 752 -48	
5 32/17*[25]	800/459 ♮G 962 -38	200/153 ♮D 464 -36	100/51 ♮A 1166 -34	25/17 ♮E 668 -32	75/68 ♮B 170 -30	
1 32/17*[5]	640/459 ♭E 575 -25	160/153 ♭B 77 -23	80/51 ♮F 779 -21	20/17 ♮C 281 -19	30/17 ♮G 983 -17	45/34 ♮D 485 -15

4.3.6 20/19* Family

19 ⸝	40/27	10/9	5/3	5/4	15/8	
	♮E	♮B	♯F	♯C	♯G	
	680	182	884	386	1088	
	-20	-18	-16	-14	-12	
5 ♮ 32/19*[25]	800/513	200/171	100/57	25/19	75/38	225/152
	♯E	♯B	𝄪F	𝄪C	𝄪G	𝄪D
	769	271	973	475	1177	679
	F-31	C-29	G-27	D-25	A-23	E-21
1 ♮ 32/19*[5]	640/513	320/171	80/57	20/19	30/19	45/38
	♯C	♯G	♯D	♯A	♯E	♯B
	383	1085	587	89	791	293
	-17	-15	-13	-11	F-9	C-7

4.3.7 40/23* Family

23 ↑	40/27	10/9	5/3	5/4	15/8	
	♮E	♮B	♯F	♯C	♯G	
	680	182	884	386	1088	
	-20	-18	-16	-14	-12	
5 ♮ 32/23*[25]	800/621	400/207	100/69	25/23		
	♮D	♮A	♮E	♮B		
	438	1140	642	144		
	D♭+38	A♭+40	E♭+42	B♭+44		
1 ♮ 32/23*[5]	640/621	320/207	80/69	40/23	30/23	45/23
	♭B	♮F	♮C	♮G	♮D	♮A
	52	754	256	958	460	1162
	-48	-46	-44	-42	-40	-38

4.4 Order 7

4.4.1 7/4* Family

23 ↑		161/81 ↑ɭA 1189 -11	161/108 ↑ɭE 691 -9	161/144 ↑ɭB 193 -7	161/96 ↑ɭ♯F 895 -5	161/128 ↑ɭ♯C 397 -3			
19 ⟋		133/81 ⟋ɭ♭G 859 -41	133/108 ⟋ɭ♭D 360 -40	133/72 ⟋ɭ♭A 1062 -38	133/96 ⟋ɭ♭E 564 -36	133/128 ⟋ɭ♭B 66 -34			
17 ⟍		119/81 ⟍ɭE 666 -34	119/108 ⟍ɭB 168 -32	119/72 ⟍ɭ♯F 870 -30	119/96 ⟍ɭ♯C 372 -28	119/64 ⟍ɭ♯G 1074 -26			
13 ɗ	364/243 ɗɭF 700 E+0	91/81 ɗɭC 202 B+2	91/54 ɗɭG 903 G♭+3	91/72 ɗɭD 405 D♭+5	91/48 ɗɭA 1107 A♭+7	91/64 ɗɭE 609 E♭+9			
11 ɫ	308/243 ɫɭ♭D 410 +10	154/81 ɫɭ♭A 1112 +12	77/54 ɫɭ♭E 614 +14	77/72 ɫɭ♭B 116 +16	77/48 ɫɭF 818 +18	77/64 ɫɭC 320 +20			
7 ɭ	392/243 ɭɭ♭G 828 F+28	98/81 ɭɭ♭D 330 C+30	49/27 ɭɭ♭A 1032 G+32	49/36 ɭɭ♭E 534 D+34	49/48 ɭɭ♭B 36 A+36	49/32 ɭɭF 738 E+38	147/128 ɭɭC 240 B+40		
5 ♮ 5/4*[7]	280/243 ɭ♮↓C 245 B+45	140/81 ɭ♮↓G 947 G♭+47	35/27 ɭ♮↓D 449 D♭+49	35/18 ɭ♮↓A 1151 -49	35/24 ɭ♮↓E 653 -47	35/32 ɭ♮↓B 155 -45	105/64 ɭ♯↓F 857 -43		
1 ♮	448/243 ɭ♭A 1059 -41	112/81 ɭ♭E 561 -39	28/27 ɭ♭B 63 -37	14/9 ɭF 765 -35	7/6 ɭC 267 -33	7/4 ɭG 969 -31	21/16 ɭD 471 -29	63/32 ɭA 1173 -27	189/128 ɭE 675 -25

4.4.2 7/5* Family

7	784/405 B 1144 A♭+44	196/135 F 645 E♭+45	49/45 C 147 B♭+47	49/30 G 849 F+49	49/40 D 351 D♭-49	147/80 A 1053 A♭-47		
5 ♮	112/81 E 561 -39	28/27 B 63 -37	14/9 F 765 -35	7/6 C 267 -33	7/4 G 969 -31	21/16 D 471 -29		
1 ♮ 8/5*[7]	448/405 C 175 B-25	224/135 G 877 -23	56/45 D 379 -21	28/15 A 1081 -19	7/5 E 583 -17	21/20 B 84 -16	63/40 F 786 -14	189/160 C 288 -12

4.4.3 14/11* Family

11	28/27 B 63 -37	14/9 F 765 -35	7/6 C 267 -33	7/4 G 969 -31	21/16 D 471 -29	
7	392/297 E 480 D-20	196/99 B 1182 A-18	49/33 F 684 E-16	49/44 C 186 B-14		
5 ♮ 20/11*[7]	560/297 A 1098 A♭-2	140/99 E 600 E♭+0	35/33 B 102 B♭+2	35/22 ♯F 804 F+4	105/88 ♯C 306 C+6	
1 ♮ 16/11*[7]	448/297 F 712 E+12	112/99 C 214 B+14	56/33 G 916 G♭+16	14/11 D 418 D♭+18	21/11 A 1119 A♭+19	63/44 E 621 E♭+21

4.4.4 14/13* Family

13	28/27 B 63, -37	14/9 F 765, -35	7/6 C 267, -33	7/4 G 969, -31	21/16 D 471, -29	
7	392/351 C 191, B-9	196/117 G 893, -7	49/39 D 395, -5	49/26 A 1097, -3		
5 20/13*[7]	560/351 F 809, +9	140/117 C 311, +11	70/39 G 1013, +13	35/26 D 515, +15	105/104 A 17, +17	
1 16/13*[7]	448/351 D 422, +22	224/117 A 1124, +24	56/39 E 626, +26	14/13 B 128, +28	21/13 F 830, +30	63/52 C 332, +32

4.4.5 28/17* Family

17	28/27 B 63, -37	14/9 F 765, -35	7/6 C 267, -33	7/4 G 969, -31	21/16 D 471, -29	
7	784/459 A 927, G♭+27	196/153 E 429, D♭+29	98/51 B 1131, A♭+31	49/34 F 633, E♭+33	147/136 C 135, B♭+35	
5 20/17*[7]	560/459 D 344, C+44	280/153 A 1046, G+46	70/51 E 548, D+48	35/34 B 50, -50	105/68 F 752, -48	
1 32/17*[7]	896/459 B 1158, A-42	224/153 F 660, E-40	56/51 C 162, B-38	28/17 G 864, -36	21/17 D 366, -34	63/34 A 1068, -32

4.4.6 28/19* Family

19 ⟋	28/27 ♭B 63 -37	14/9 F 765 -35	7/6 C 267 -33	7/4 G 969 -31	21/16 D 471 -29	
5 ♮	560/513 ♮B 152 -48	280/171 ♯F 854 -46	70/57 ♯C 356 -44	35/19 ♯G 1058 -42	105/76 ♯D 560 -40	315/304 ♯A 62 -38
1 ♮	896/513 G 965 -35	224/171 D 467 -33	112/57 A 1169 -31	28/19 E 671 -29	21/19 B 173 -27	63/38 ♯F 875 -25

4.4.7 28/23* Family

23 ↑	28/27 ♭B 63 -37	14/9 F 765 -35	7/6 C 267 -33	7/4 G 969 -31	21/16 D 471 -29	
5 ♮	1120/621 ♭A 1021 G+21	280/207 ♭E 523 D+23	70/69 ♭B 25 A+25	35/23 ♮F 727 E+27	105/92 ♮C 229 B+29	
1 ♮	896/621 ♭F 635 E♭+35	224/207 ♭C 137 B♭+37	112/69 ♭G 839 F+39	28/23 ♭D 341 C+41	42/23 ♭A 1043 G+43	63/46 ♭E 544 D+44

5. Tonal network (brass)

Following the same template established for the tonal network in chapter 4, chapter 5 presents a tonal network based on a *B♭* generating pitch. This network befits brass instruments more naturally[1], since these are usually double instruments in *B♭* and *F*[2]. The instrument should be tuned to *B♭* +0 (no valve, "*B♭* 0" fingering), while the transposing valve should be adjusted to subtract exactly the 4/3 interval to produce *F* +2 ("F 0" fingering). To these reference notes, we add the ones produced by valve 1, usually adjusted to subtract exactly the 9/8 interval to produce *E♭* -2 on an *F* instrument ("F 1" fingering) and *A♭* -4 on a *B♭* instrument ("B♭ 1" fingering)[3]. Similarly to string instruments, each note is thus produced in the 3-identity of its lower neighbour. For example, let us have a look at these four French horn fingerings:

fingering	ratio	note
F 0	3/2	*F* +2
B♭ 0	1/1	*B♭* +0
F 1	4/3	*E♭* -2
B♭ 1	16/9	*A♭* -4

Table 5.1: Values of the vibration of four French horn fingerings in fundamental mode

If, in a table that presents this information as denominatives, we organize the multiples of 3 horizontally, all there is left to do is to organize the 1-, 5-, and 7-harmonics vertically.

	B♭ 1 fingering	F 1 fingering	B♭ 0 fingering	F 0 fingering	
7-harmonic	14/9	7/6	7/4	21/16	
5-harmonic	10/9	5/3	5/4	15/8	
1-harmonic	16/9	4/3	1/1	3/2	9/8

Table 5.2: French horn reference notes expressed as denominatives

[1] See Hayward and Sabat, 2006.

[2] This means that the instrument is equipped with the equivalent of two valve-less air columns, akin to two open strings, whose fundamentals are *B♭* and *F*, respectively. Switching between these columns is done with a transposition valve, while the other valves change the pitch produced by one of these columns.

[3] See Hayward and Sabat, 2006.

Thus the ratios of the tonal network of brass instruments are identical to the ratios of string instruments transposed by 3/2. The thirteen reference notes of brass instruments are:

7	14/9 L♭G 765 -35	7/6 L♭D 267 -33	7/4 L♭A 969 -31	21/16 L♭E 471 -29	
5	10/9 ♮↓C 182 -18	5/3 ♮↓G 884 -16	5/4 ♮↓D 386 -14	15/8 ♮↓A 1088 -12	
1	16/9 ♭A 996 -4	4/3 ♭E 498 -2	1/1 ♭B 0 +0	3/2 ♮F 702 +2	9/8 ♮C 204 +4

Table 5.3: Values of the thirteen references notes for a tuning fork at *B♭* = 1/1

When used together with string instruments, brass instruments can be tuned to *B♭* +12[4], instead of *B♭* +0. When doing so,

— the reference notes produced by the 1-harmonic of the brass instruments correspond to the ones produced by the string instruments for the 8/5* family of tonalities;

256/135 ↑♭A 1108 +8	64/45 ↑♭E 610 +10	16/15 ↑♭B 112 +12	8/5 ↑♮F 814 +14	6/5 ↑♮C 316 +16

Table 5.4: 1-harmonics of a brass instrument tuned to *B♭* +12

— consequently, the reference notes produced by the 5-harmonic of brass instruments correspond to the ones produced by string instruments for the 1/1* family of tonalities.

32/27 ♮C 294 -6	16/9 ♮G 996 -4	4/3 ♮D 498 -2	1/1 ♮A 0 +0	3/2 ♮E 702 +2

Table 5.5: 5-harmonics of a brass instrument tuned to *B♭* +12

[4] See Hayward and Sabat, 2006.

5.1 Summary table

	/1	5/	7/
	27/16* 9/8* 3/2* 1/1* 4/3* 16/9* 32/27* 128/81*	- 45/32* 15/8* 5/4* 5/3* 10/9* 40/27* 160/81*	- - 21/16* 7/4* 7/6* 14/9* 28/27* 112/81*
/5	27/20* 9/5* 6/5* 8/5* 16/15* 64/45*		- - 21/20* 7/5* 28/15* 56/45*
/7	24/17* 9/7* 12/7* 8/7* 32/21* 64/63*	- 45/28* 15/14* 10/7* 40/21* 80/63*	
/11	18/11* 12/11* 16/11* 64/33* 128/99*	- 15/11* 20/11* 40/33* 160/99*	- 21/11* 14/11* 56/33* 112/99*
/13	18/13* 24/13* 16/13* 64/39* 128/117*	- 15/13* 20/13* 40/39* 160/117*	- 21/13* 14/13* 56/39* 224/117*
/17	18/17* 24/17* 32/17* 64/51* 256/153*	- 30/17* 20/17* 80/51* 160/153*	- 21/17* 28/17* 56/51* 224/153*
/19	36/19* 24/19* 32/19* 64/57* 128/171*	- 30/19* 20/19* 80/57* 320/171*	- 21/19* 28/19* 112/57* 224/171*
/23	36/23* 24/23* 32/23* 128/69* 512/267*	- 30/23* 40/23* 80/69* 320/207*	- 42/23* 28/23* 112/69* 448/267*

5.2.1 1/1* Family

23 ↑		46/27 ↑G 922 +22	23/18 ↑D 424 +24	23/12 ↑A 1126 +26	23/16 ↑E 628 +28	69/64 ↑B 130 +30	207/128 ↑♯F 832 +32	621/512 ↑♯C 334 +34		
19 ⁄	152/81 ⁄𝄫B 1090 A-10	38/27 ⁄♭F 592 E-8	19/18 ⁄♭C 94 B-6	19/12 ⁄♭G 796 -4	19/16 ⁄♭D 298 -2	57/32 ⁄♭A 999 -1	171/128 ⁄♭E 501 +1	513/512 ⁄♭B 3 +3		
17 ⧹	136/81 ⧹G 897 -3	34/27 ⧹D 399 -1	17/9 ⧹A 1101 +1	17/12 ⧹E 603 +3	17/16 ⧹B 105 +5	51/32 ⧹♯F 807 +7	153/128 ⧹♯C 309 +9	459/256 ⧹♯G 1011 +11		
13 𝄳	104/81 𝄳♭E 433 D+33	52/27 𝄳♭B 1135 A+35	13/9 𝄳F 637 E+37	13/12 𝄳C 139 B+39	13/8 𝄳G 841 G♭+41	39/32 𝄳D 342 D♭+42	117/64 𝄳A 1044 A♭+44	351/256 𝄳E 546 E♭+46		
11 𝄲	88/81 𝄲♭C 143 B+43	44/27 𝄲♭G 845 +45	11/9 𝄲♭D 347 +47	11/6 𝄲♭A 1049 +49	11/8 𝄲♭E 551 E-49	33/32 𝄲♭B 53 B-47	99/64 𝄲F 755 F♯-45	297/256 𝄲C 257 C♯-43		
7 ˪	112/81 ˪♭F 561 E-39	28/27 ˪♭C 63 B-37	14/9 ˪♭G 765 -35	7/6 ˪♭D 267 -33	7/4 ˪♭A 969 -31	21/16 ˪♭E 471 -29	63/32 ˪♭B 1173 -27	189/128 ˪F 675 -25		
5 ♮↓ 8/5*[25]	160/81 ♭↓B 1178 -22	40/27 ♮↓F 680 -20	10/9 ♮↓C 182 -18	5/3 ♮↓G 884 -16	5/4 ♮↓D 386 -14	15/8 ♮↓A 1088 -12	45/32 ♮↓E 590 -10	135/128 ♮↓B 92 -8		
1 ♮	128/81 ♭G 792 -8	32/27 ♭D 294 -6	16/9 ♭A 996 -4	4/3 ♭E 498 -2	1/1 ♭B 0 +0	3/2 ♮F 702 +2	9/8 ♮C 204 +4	27/16 ♮G 906 +6	81/64 ♮D 408 +8	243/128 ♮A 1110 +10

5.2.2 8/5* Family

23 ↑		46/45 ↑♭B 38 +38	23/15 ↑♮F 740 +40	23/20 ↑♮C 242 +42	69/40 ↑♮G 944 +44	207/160 ↑♮D 446 +46	621/320 ↑♮A 1148 +48		
19 ⸝	152/135 ⸝𝄫D 205 C+5	76/45 ⸝𝄫A 907 G+7	19/15 ⸝𝄫E 409 D+9	19/10 ⸝𝄫B 1111 A+11	57/40 ⸝♭F 613 E+13	171/160 ⸝♭C 115 B+15	513/320 ⸝♭G 817 +17		
17 ≈		68/45 ≈♮F 715 +15	17/15 ≈♮C 217 +17	17/10 ≈♮G 919 +19	51/40 ≈♮D 421 +21	153/80 ≈♮A 1123 +23	459/320 ≈♮E 625 +25		
13 ꝺ		52/45 ꝺ♭D 250 -50	26/15 ꝺ♭A 952 -48	13/10 ꝺ♭E 454 -46	39/20 ꝺ♭B 1156 -44	117/80 ꝺ♮F 658 -42	351/320 ꝺ♮C 160 -40		
11 +		88/45 +𝄫B 1161 B♭-39	22/15 +♭F 663 F-37	11/10 +♭C 165 C-35	33/20 +♭G 867 G-33	98/80 +♭D 369 D-31	297/160 +♭A 1071 A-29		
7 L 7/5*[1]	224/135 L𝄫A 877 G-23	56/45 L𝄫E 379 D-21	28/15 L𝄫B 1081 A-19	7/5 L♭F 583 E-17	21/20 L♭C 84 B-16	63/40 L♭G 786 -14	189/160 L♭D 288 -12		
5 ♮	32/27 ♭D 294 -6	16/9 ♭A 996 -4	4/3 ♭E 498 -2	1/1 ♭B 0 +0	3/2 ♮F 702 +2	9/8 ♮C 204 +4	27/16 ♮G 906 +6		
1 ♮	256/135 𝄫B 1108 A+8	64/45 ♭F 610 E+10	16/15 ♭C 112 B+12	8/5 ♭G 814 +14	6/5 ♭D 316 +16	9/5 ♭A 1018 +18	27/20 ♭E 520 +20	81/80 ♭B 22 +22	243/160 ♮F 723 +23

5.2.3 8/7* Family

17	68/63 B 132 +32	34/21 ♯F 834 +34	17/14 ♯C 336 +36	51/28 ♯G 1038 +38	153/112 ♯D 540 +40	459/448 ♯A 42 +42		
13	104/63 G 868 -32	26/21 D 370 -30	13/7 A 1072 -28	39/28 E 574 -26	117/112 B 76 -24	351/224 ♯F 778 -22		
11	88/63 ♭E 579 E-21	22/21 ♭B 81 B-19	11/7 F 782 F♯-18	33/28 C 284 C♯-16	99/56 G 986 G♯-14	297/224 D 488 D♯-12		
7	16/9 ♭A 996 -4	4/3 ♭E 498 -2	1/1 ♭B 0 +0	3/2 ♮F 702 +2	9/8 ♮C 204 +4	27/16 ♮G 906 +6		
5 ♮ 10/7*[1]	80/63 ♮D 414 +14	40/21 ♮A 1116 +16	10/7 ♮E 617 +17	15/14 ♮B 119 +19	45/28 ♯F 821 +21	135/112 ♯C 323 +23		
1 ♮	64/63 ♭B 27 +27	32/21 F 729 +29	8/7 C 231 +31	12/7 G 933 +33	9/7 D 435 +35	27/14 A 1137 +37	81/56 E 639 +39	243/224 B 141 +41

5.2.4 16/11* Family

11 ɫ	16/9 ♭A 996 -4	4/3 ♭E 498 -2	1/1 ♭B 0 +0	3/2 ♮F 702 +2	9/8 ♮C 204 +4	27/16 ♮G 906 +6	
7 ʟ 14/11*[1]	112/99 dʟ♭D 214 C+14	56/33 dʟ♭A 916 G+16	14/11 dʟ♭E 418 D+18	21/11 dʟ♭B 1119 A+19	63/44 dʟF 621 E+21	189/176 dʟC 123 B+23	
5 ♮↓ 20/11*[1]	160/99 d♮↓G 831 G♭+31	40/33 d♮↓D 333 D♭+33	20/11 d♮↓A 1035 A♭+35	15/11 d♮↓E 537 E♭+37	45/44 d♮↓B 39 B♭+39	135/88 d♯↓F 741 F+41	405/352 d♯↓C 243 C+43
1 ♮	128/99 d♭E 445 D+45	64/33 d♭B 1147 A+47	16/11 dF 649 E+49	12/11 dC 151 -49	18/11 dG 853 -47	27/22 dD 355 -45	81/44 dA 1057 -43

5.2.5 16/13* Family

13 ⅆ	16/9 ♭A 996 -4	4/3 ♭E 498 -2	1/1 ♭B 0 +0	3/2 ♮F 702 +2	9/8 ♮C 204 +4	27/16 ♮G 906 +6	
7 ʟ 14/13*[1]	224/117 ‡ʟ♭♭B 1124 A+24	56/39 ‡ʟ♭F 626 E+26	14/13 ‡ʟ♭C 128 B+28	21/13 ‡ʟ♭G 830 +30	63/52 ‡ʟ♭D 332 +32	189/104 ‡ʟ♭A 1034 +34	
5 ♮↓ 20/13*[1]	160/117 ‡♭↓E 542 +42	40/39 ‡♭↓B 44 +44	20/13 ‡♮↓F 746 +46	15/13 ‡♮↓C 248 +48	45/26 ‡♮↓G 950 +50	135/104 ‡♮↓D 452 D♯-48	405/208 ‡♮↓A 1154 A♯-46
1 ♮	128/117 ‡♭C 156 C-44	64/39 ‡♭G 858 G-42	16/13 ‡♭D 359 D-41	24/13 ‡♭A 1061 A-39	18/13 ‡♭E 563 E-37	27/26 ‡♭B 65 B-35	81/52 ‡F 767 F♯-33

5.2.6 32/17* Family

17	16/9 ♭A 996 -4	4/3 ♭E 498 -2	1/1 ♭B 0 +0	3/2 ♮F 702 +2	9/8 ♮C 204 +4	27/16 ♮G 906 +6	
7 28/17*[1]	224/153 ♭♭G 660 F-40	56/51 ♭♭D 162 C-38	28/17 ♭♭A 864 G-36	21/17 ♭♭E 366 D-34	63/34 ♭♭B 1068 A-32	189/136 ♭F 570 E-30	
5 ♮ 20/17*[1]	160/153 ♭C 77 B-23	80/51 ♭G 779 -21	20/17 ♭D 281 -19	30/17 ♭A 983 -17	45/34 ♭E 485 -15	135/68 ♭B 1187 -13	
1 ♮	256/153 ♭♭A 891 G-9	64/51 ♭♭E 393 D-7	32/17 ♭♭B 1095 A-5	24/17 ♭F 597 E-3	18/17 ♭C 99 B-1	27/17 ♭G 801 +1	81/68 ♭D 303 +3

5.2.7 32/19* Family

19	16/9 ♭A 996 -4	4/3 ♭E 498 -2	1/1 ♭B 0 +0	3/2 ♮F 702 +2	9/8 ♮C 204 +4	27/16 ♮G 906 +6		
5 ♮ 20/19*[1]	320/171 ♮A 1085 -15	80/57 ♮E 587 -13	20/19 ♮B 89 -11	30/19 ♯F 791 -9	45/38 ♯C 293 -7	135/76 ♯G 995 -5	405/304 ♯D 497 -3	
1 ♮	256/171 F 699 -1	64/57 C 201 +1	32/19 G 902 +2	24/19 D 404 +4	36/19 A 1106 +6	27/19 E 608 +8	81/76 B 110 +10	243/152 ♯F 812 +12

5.2.8 32/23* Family

23 ↑	16/9 ♭A 996 -4	4/3 ♭E 498 -2	1/1 ♭B 0 +0	3/2 ♮F 702 +2	9/8 ♮C 204 +4	27/16 ♮G 906 +6	
5 ♮↓ 40/23*[1]	320/207 ↓↓♭G 754 -46	80/69 ↓↓♭D 256 -44	40/23 ↓↓♭A 958 -42	30/23 ↓↓♭E 460 -40	45/23 ↓↓♭B 1162 -38	135/92 ↓↓♮F 664 -36	
1 ♮	256/207 ↓𝄫E 368 D-32	128/69 ↓𝄫B 1070 A-30	32/23 ↓♭F 572 E-28	24/23 ↓♭C 74 B-26	36/23 ↓♭G 776 -24	27/23 ↓♭D 278 -22	81/46 ↓♭A 980 -20

5.3 Order 5

5.3.1 5/4* Family

23		115/108 ♮B 109 +9	115/72 ♯F 811 +11	115/96 ♯C 313 +13	115/64 ♯G 1015 +15	345/256 ♯D 517 +17			
19		95/54 ♭A 978 -22	95/72 ♭E 480 -20	95/48 ♭B 1182 -18	95/64 ♮F 684 -16	285/256 ♮C 186 -14	855/512 ♮G 888 -12		
17		85/54 ♯F 785 -15	85/72 ♯C 287 -13	85/48 ♯G 989 -11	85/64 ♯D 491 -9	255/128 ♯A 1193 -7			
13	130/81 ♮G 819 G♭+19	65/54 ♮D 321 D♭+21	65/36 ♮A 1023 A♭+23	65/48 ♮E 525 E♭+25	65/64 ♮B 27 B♭+27	195/128 ♯F 729 F+29			
11	110/81 ♭E 530 +30	55/54 ♭B 32 +32	55/36 ♮F 734 +34	55/48 ♮C 236 +36	55/32 ♮G 938 +38	165/128 ♮D 440 +40			
7 7/4*[5]	140/81 ♭A 947 G+47	35/27 ♭E 449 D+49	35/18 ♭B 1151 -49	35/24 ♮F 653 -47	35/32 ♮C 155 -45	105/64 ♮G 857 -43	315/256 ♮D 359 -41		
5 ♮ 1/1*[25]	100/81 ♮D 365 -35	50/27 ♮A 1067 -33	25/18 ♮E 569 -31	25/24 ♮B 71 -29	25/16 ♯F 773 -27	75/64 ♯C 275 -25	225/128 ♯G 977 -23		
1 ♮ 8/5*[25]	160/81 ♭B 1178 -22	40/27 ♮F 680 -20	10/9 ♮C 182 -18	5/3 ♮G 884 -16	5/4 ♮D 386 -14	15/8 ♮A 1088 -12	45/32 ♮E 590 -10	135/128 ♮B 92 -8	405/256 ♯F 794 -6

5.3.2 10/7* Family

7	10/9 ♮C 182 -18	5/3 ♮G 884 -16	5/4 ♮D 386 -14	15/8 ♮A 1088 -12	45/32 ♮E 590 -10		
5 ♮ 8/7*[25]	100/63 ♯F 800 +0	25/21 ♯C 302 +2	25/14 ♯G 1004 +4	75/56 ♯D 506 +6	225/128 ♯A 977 +7		
1 ♮ 8/7*[5]	80/63 ♮D 414 +14	40/21 ♮A 1116 +16	10/7 ♮E 617 +17	15/14 ♮B 119 +19	45/28 ♯F 821 +21	135/112 ♯C 323 +23	405/224 ♯G 1025 +25

5.3.3 20/11* Family

11	10/9 ♮C 182 -18	5/3 ♮G 884 -16	5/4 ♮D 386 -14	15/8 ♮A 1088 -12	45/32 ♮E 590 -10		
7 14/11*[5]	140/99 ♮F 600 E+0	35/33 ♮C 102 B+2	35/22 ♮G 804 G♭+4	105/88 ♮D 306 D♭+6	315/176 ♮A 1008 A♭+8		
5 ♮ 16/11*[25]	100/99 ♮B 17 B♭+17	50/33 ♯F 719 F+19	25/22 ♯C 221 C+21	75/44 ♯G 923 G+23	225/176 ♯D 425 D+25		
1 ♮ 16/11*[5]	160/99 ♮G 831 G♭+31	40/33 ♮D 333 D♭+33	20/11 ♮A 1035 A♭+35	15/11 ♮E 537 E♭+37	45/44 ♮B 39 B♭+39	135/88 ♯F 741 F+41	405/352 ♯C 243 C+43

5.3.4 20/13* Family

13	10/9 ♮C 182 -18	5/3 ♮G 884 -16	5/4 ♮D 386 -14	15/8 ♮A 1088 -12	45/32 ♮E 590 -10		
7 14/13*[5]	140/117 ♯♭D 311 +11	70/39 ♯♭A 1013 +13	35/26 ♯♭E 515 +15	105/104 ♯♭B 17 +17	315/208 ♯♮F 719 +19		
5 ♮ 16/13*[25]	200/117 ♯♮G 928 +28	50/39 ♯♮D 430 +30	25/13 ♯♮A 1132 +32	75/52 ♯♮E 634 +34	225/208 ♯♮B 136 +36		
1 ♮ 16/13*[5]	160/117 ♯♭E 542 +42	40/39 ♯♭B 44 +44	20/13 ♯♮F 746 +46	15/13 ♯♮C 248 +48	45/26 ♯♮G 950 +50	135/104 ♯♮D 452 D♯-48	405/208 ♯♮A 1154 A♯-46

5.3.5 20/17* Family

17	10/9 ♮C 182 -18	5/3 ♮G 884 -16	5/4 ♮D 386 -14	15/8 ♮A 1088 -12	45/32 ♮E 590 -10	
7 28/17*[5]	280/153 𝄫B 1046 A♭+46	70/51 ♭F 548 E♭+48	35/34 ♭C 50 B-50	105/68 ♭G 752 -48	315/272 ♭D 752 -46	
5 ♮ 32/17*[25]	200/153 ♭E 464 -36	100/51 ♭B 1166 -34	25/17 ♮F 668 -32	75/68 ♮C 170 -30	225/136 ♮G 872 -28	
1 ♮ 32/17*[5]	160/153 ♭C 77 B-23	80/51 ♭G 779 -21	20/17 ♭D 281 -19	30/17 ♭A 983 -17	45/34 ♭E 485 -15	135/68 ♭B 1187 -13

5.3.6 20/19* Family

19	10/9 ♮C 182 -18	5/3 ♮G 884 -16	5/4 ♮D 386 -14	15/8 ♮A 1088 -12	45/32 ♮E 590 -10	
5 ♮ 32/19*[25]	200/171 ♯C 271 -29	100/57 ♯G 973 -27	25/19 ♯D 475 -25	75/38 ♯A 1177 -23	225/152 ♯E 679 F-21	675/608 ♯B 181 C-19
1 ♮ 32/19*[5]	320/171 ♮A 1085 -15	80/57 ♮E 587 -13	20/19 ♮B 89 -11	30/19 ♯F 791 -9	45/38 ♯C 293 -7	135/76 ♯G 995 -5

5.3.7 40/23* Family

23 ↑	10/9 ♮C 182 -18	5/3 ♮G 884 -16	5/4 ♮D 386 -14	15/8 ♮A 1088 -12	45/32 ♮E 590 -10	
5 ♮ 32/23*[25]	400/207 ♭B 1140 A+40	100/69 ♮F 642 E+42	25/23 ♮C 144 B+44	75/46 ♮G 846 G♭+46		
1 ♮ 32/23*[5]	320/207 ♭G 754 -46	80/69 ♭D 256 -44	40/23 ♭A 958 -42	30/23 ♭E 460 -40	45/23 ♭B 1162 -38	135/92 ♮F 664 -36

5.4 Order 7

5.4.1 7/4* Family

23		161/108 F 691 -9	161/144 C 193 -7	161/96 G 895 -5	161/128 D 397 -3	483/256 A 1099 -1			
19		133/108 E 360 D-40	133/72 B 1062 A-38	133/96 F 564 E-36	133/128 C 66 B-34	399/256 G 768 -32			
17		119/108 C 168 -32	119/72 G 870 -30	119/96 D 372 -28	119/64 A 1074 -26	357/256 E 576 -24			
13	91/81 D 202 C+2	91/54 A 903 G+3	91/72 E 405 D+5	91/48 B 1107 A+7	91/64 F 609 E+9	273/256 C 111 B+11			
11	154/81 B 1112 A+12	77/54 F 614 E+14	77/72 C 116 B+16	77/48 G 818 +18	77/64 D 320 +20	231/128 A 1022 +22			
7	98/81 E 330 D♭+30	49/27 B 1032 A♭+32	49/36 F 534 E♭+34	49/48 C 36 B♭+36	49/32 G 738 F+38	147/128 D 240 C+40	441/256 A 942 G+42		
5 5/4*[7]	140/81 A 947 G+47	35/27 E 449 D+49	35/18 B 1151 -49	35/24 F 653 -47	35/32 C 155 -45	105/64 G 857 -43	315/256 D 359 -41		
1	112/81 F 561 E-39	28/27 C 63 B-37	14/9 G 765 -35	7/6 D 267 -33	7/4 A 969 -31	21/16 E 471 -29	63/32 B 1173 -27	189/128 F 675 -25	567/512 C 177 -23

5.4.2 7/5* Family

7	196/135 G 645 E+45	49/45 D 147 B+47	49/30 A 849 G♭+49	49/40 E 351 D-49	147/80 B 1053 A-47	441/320 F 555 E-45		
5	28/27 C 63 B-37	14/9 G 765 -35	7/6 D 267 -33	7/4 A 969 -31	21/16 E 471 -29	63/32 B 1173 -27		
1 8/5*[7]	224/135 A 877 G-23	56/45 E 379 D-21	28/15 B 1081 A-19	7/5 F 583 E-17	21/20 C 84 B-16	63/40 G 786 -14	189/160 D 288 -12	567/320 A 990 -10

5.4.3 14/11* Family

11	14/9 G 765 -35	7/6 D 267 -33	7/4 A 969 -31	21/16 E 471 -29	63/32 B 1173 -27	
7	196/99 C 1182 B♭-18	49/33 G 684 F-16	49/44 D 186 C-14	147/88 A 888 G-12		
5 20/11*[7]	140/99 F 600 E+0	35/33 C 102 B+2	35/22 G 804 G♭+4	105/88 D 306 D♭+6	315/176 A 1008 A♭+8	
1 16/11*[7]	112/99 D 214 C+14	56/33 A 916 G+16	14/11 E 418 D+18	21/11 B 1119 A+19	63/44 F 621 E+21	189/176 C 123 B+23

5.4.4 14/13* Family

13	14/9 ♭G 765 -35	7/6 ♭D 267 -33	7/4 ♭A 969 -31	21/16 ♭E 471 -29	63/32 ♭B 1173 -27	
7	196/117 ♭♭A 893 G-7	49/39 ♭♭E 395 D-5	49/26 ♭♭B 1097 A-3	147/104 ♭F 599 E-1		
5 20/13*[7]	140/117 ♭D 311 +11	70/39 ♭A 1013 +13	35/26 ♭E 515 +15	105/104 ♭B 17 +17	315/208 F 719 +19	
1 16/13*[7]	224/117 ♭♭B 1124 A+24	56/39 ♭F 626 E+26	14/13 ♭C 128 B+28	21/13 ♭G 830 +30	63/52 ♭D 332 +32	189/104 ♭A 1034 +34

5.4.5 28/17* Family

17	14/9 ♭G 765 -35	7/6 ♭D 267 -33	7/4 ♭A 969 -31	21/16 ♭E 471 -29	63/32 ♭B 1173 -27	
7	196/153 ♭♭F 429 D+29	98/51 ♭♭C 1131 A+31	49/34 ♭♭G 633 E+33	147/136 ♭♭D 135 B+35	441/272 ♭♭A 837 G+37	
5 20/17*[7]	280/153 ♭♭B 1046 A♭+46	70/51 ♭F 548 E♭+48	35/34 ♭C 50 B-50	105/68 ♭G 752 -48	315/272 ♭D 254 -46	
1 32/17*[7]	224/153 ♭♭G 660 F-40	56/51 ♭♭D 162 C-38	28/17 ♭♭A 864 G-36	21/17 ♭♭E 366 D-34	63/34 ♭♭B 1068 A-32	189/136 ♭F 570 E-30

5.4.6 28/19* Family

19	14/9 ♭G 765 -35	7/6 ♭D 267 -33	7/4 ♭A 969 -31	21/16 ♭E 471 -29	63/32 ♭B 1173 -27	
5 ♮↓	280/171 ♮G 854 -46	70/57 ♮D 356 -44	35/19 ♮A 1058 -42	105/76 ♮E 560 -40	315/304 ♮B 62 -38	945/608 ♯F 763 -37
1 ♮	224/171 ♭E 467 -33	112/57 ♭B 1169 -31	28/19 F 671 -29	21/19 C 173 -27	63/38 G 875 -25	189/152 D 377 -23

5.4.7 28/23* Family

23 ↑	14/9 ♭G 765 -35	7/6 ♭D 267 -33	7/4 ♭A 969 -31	21/16 ♭E 471 -29	63/32 ♭B 1173 -27	
5 ♮↓	280/207 ↓♭F 523 E♭+23	70/69 ↓♭C 25 B♭+25	35/23 ↓♭G 727 F+27	105/92 ↓♭D 229 C+29	315/184 ↓♭A 931 G+31	
1 ♮	224/207 ↓𝄫D 137 B+37	112/69 ↓𝄫A 839 G♭+39	28/23 ↓𝄫E 341 D♭+41	42/23 ↓𝄫B 1043 A♭+43	63/46 ↓♭F 544 E♭+44	189/184 ↓♭C 46 B♭+46

6. Tonality pairs

This is the summary table of the main modulation factors used in the *Supplements* to the *Treatise*. The simplest factors, which we recommend should be used most often, are boxed below. Arranged in ascending order, they serve as titles for the subsections of this chapter.

*16/15 (112 c)	*15/8 (1088 c)
*15/14 (119 c)	*28/15 (1081 c)
*10/9 (182 c)	*9/5 (1018 c)
*9/8 (204 c)	*16/9 (996 c)
*8/7 (231 c)	*7/4 (969 c)
*7/6 (267 c)	*12/7 (933 c)
*75/64 (275 c)	*128/75 (925 c)
*32/27 (294 c)	*27/16 (906 c)
*6/5 (316 c)	*5/3 (884 c)
*5/4 (386 c)	*8/5 (814 c)
*81/64 (408 c)	*128/81 (792 c)
*32/25 (427 c)	*25/16 (773 c)
*9/7 (435 c)	*14/9 (765 c)
*21/16 (471 c)	*32/21 (729 c)
*4/3 (498 c)	*3/2 (702 c)
*27/20 (520 c)	*40/27 (680 c)
*7/5 (583 c)	*10/7 (617 c)
*45/32 (590 c)	*64/45 (610 c)

Table 6.1: Main modulation factors

The tables found in the next pages present the tonality pairs corresponding to each of these factors. Tonality pairs are classified in three columns according to the order 1, 5, or 7 of the starting tonality, then, from top to bottom in each column by multiples of 3. Inside each column, pair groups are classified from top to bottom according to the following rules:

— the simplest generating harmonic of the starting tonality;
— the simplest generating harmonic of the destination tonality;
— the simplest identity of the starting harmonic;
— the simplest identity of the destination harmonic.

For example, here is the table for the *10/7 modulation factor (for which there is no order 5 starting tonality). Pairs in grey, like 27/20* → 27/14*, involve tonalities located at the limits of the tonal network (see chapter 4).

*10/7 (617 c)

27/20* → 27/14*		21/20* → 3/2*
9/5* → 9/7*		7/5* → 1/1*
6/5* → 12/7*		28/15* → 4/3*
8/5* → 8/7*		56/45* → 16/9*
16/15* → 32/21*		224/135* → 32/27*
64/45* → 64/63*		
256/135* → 256/189*		21/16* → 15/8*
		7/4* → 5/4*
9/8* → 45/28*		7/6* → 5/3*
3/2* → 15/14*		14/9* → 10/9*
1/1* → 10/7*		28/27* → 40/27*
4/3* → 40/21*		112/81* → 160/81*
16/9* → 80/63*		448/243* → 320/243*
32/27* → 320/189*		

Table 6.2: *10/7 (617 c) modulation factor and corresponding tonality pairs

The order in which tonalities are presented is generally important: it is better to use the tonalities that stray the least from the generating pitch in order to encourage tonal coherence and make composing and performing easier. For instance, referring to the above table, we prefer: 8/5* → 8/7* to 1/1* → 10/7*; 1/1* → 10/7* to 7/5* → 1/1*; etc.

6.1 Factor *16/15 (112 c)

*16/15 (112 c)

27/16* → 27/20* 9/8* → 9/5* 3/2* → 8/5* 1/1* → 16/15* 4/3* → 64/45* 16/9* → 256/135*	45/32* → 3/2* 15/8* → 1/1* 5/4* → 4/3* 5/3* → 16/9* 10/9* → 32/27* 40/27* → 128/81* 160/81* → 256/243* 45/28* → 12/7* 15/14* → 8/7* 10/7* → 32/21* 40/21* → 64/63* 80/63* → 256/189*	21/16* → 7/5* 7/4* → 28/15* 7/6* → 56/45* 14/9* → 224/135*

*15/14 (119 c)

9/5* → 27/14* 6/5* → 9/7* 8/5* → 12/7* 16/15* → 8/7* 64/45* → 32/21* 256/135* → 64/63* 3/2* → 45/28* 1/1* → 15/14* 4/3* → 10/7* 16/9* → 40/21* 32/27* → 80/63* 128/81* → 320/189*		21/20* → 9/8* 7/5* → 3/2* 28/15* → 1/1* 56/45* → 4/3* 224/135* → 16/9* 21/16* → 45/32* 7/4* → 15/8* 7/6* → 5/4* 14/9* → 5/3* 28/27* → 10/9* 112/81* → 40/27* 448/243* → 160/81*

6.2 Factor *10/9 (182 c)

*10/9 (182 c)

27/20* → 3/2* 9/5* → 1/1* 6/5* → 4/3* 8/5* → 16/9* 16/15* → 32/27* 64/45* → 128/81* 256/135* → 256/243* 27/16* → 15/8* 9/8* → 5/4* 3/2* → 5/3* 1/1* → 10/9* 4/3* → 40/27* 16/9* → 160/81* 32/27* → 320/243* 27/14* → 45/28* 9/7* → 10/7* 12/7* → 40/21* 8/7* → 80/63* 32/21* → 320/189*		21/20* → 7/6* 7/5* → 14/9* 28/15* → 28/27* 56/45* → 112/81* 224/135* → 448/243*

6.3 Factor *9/8 (204 c)

*9/8 (204 c)

3/2* → 27/16* 1/1* → 9/8* 4/3* → 3/2* 16/9* → 1/1* 32/27* → 4/3* 128/81* → 16/9* 256/243* → 32/27* 9/5* → 27/20* 8/5* → 9/5* 16/15* → 6/5* 64/45* → 8/5* 256/135* → 16/15* 12/7* → 27/14* 8/7* → 9/7* 32/21* → 12/7* 64/63* → 8/7* 256/189* → 32/21*	10/9* → 5/4* 40/27* → 5/3* 160/81* → 10/9* 320/243* → 40/27* 10/7* → 45/28* 40/21* → 15/14* 80/63* → 10/7* 320/189* → 40/21*	7/6* → 21/16* 14/9* → 7/4* 28/27* → 7/6* 112/81* → 14/9* 448/243* → 28/27* 28/15* → 21/20* 56/45* → 7/5* 224/135* → 28/15*

6.4 Factor *8/7 (231 c)

*8/7 (231 c)

27/16* → 27/14* 9/8* → 9/7* 3/2* → 12/7* 1/1* → 8/7* 4/3* → 32/21* 16/9* → 64/63* 32/27* → 256/189*	45/32* → 45/28* 15/8* → 15/14* 5/4* → 10/7* 5/3* → 40/21* 10/9* → 80/63* 40/27* → 320/189*	21/16* → 3/2* 7/4* → 1/1* 7/6* → 4/3* 14/9* → 16/9* 28/27* → 32/27* 112/81* → 128/81* 448/243* → 256/243* 21/20* → 6/5* 7/5* → 8/5* 28/15* → 16/15* 56/45* → 64/45* 224/135* → 256/135*

6.5 Factor *7/6 (267 c)

*7/6 (267 c)

27/14* → 9/8* 9/7* → 3/2* 12/7* → 1/1* 8/7* → 4/3* 32/21* → 16/9* 64/63* → 32/27* 256/189* → 128/81* 9/8* → 21/16* 3/2* → 7/4* 1/1* → 7/6* 4/3* → 14/9* 16/9* → 28/27* 32/27* → 112/81* 128/81* → 448/243* 9/5* → 21/20* 6/5* → 7/5* 8/5* → 28/15* 16/15* → 56/45* 64/45* → 224/135*	45/28* → 15/8* 15/14* → 5/4* 10/7* → 5/3* 40/21* → 10/9* 80/63* → 40/27* 320/189* → 160/81*	

*75/64 (275 c)

6/5* → 45/32* 8/5* → 15/8* 16/15* → 5/4* 64/45* → 5/3* 256/135* → 10/9*		

*32/27 (294 c)

27/16* → 1/1* 9/8* → 4/3* 3/2* → 16/9* 1/1* → 32/27* 4/3* → 128/81* 16/9* → 256/243* 27/20* → 8/5* 9/5* → 16/15* 6/5* → 64/45* 8/5* → 256/135* 9/7* → 32/21* 12/7* → 64/63* 8/7* → 256/189*	45/32* → 5/4* 15/8* → 10/9* 5/4* → 40/27* 5/3* → 160/81* 10/9* → 320/243* 45/28* → 40/21* 15/14* → 80/63* 10/7* → 320/189*	21/16* → 14/9* 7/4* → 28/27* 7/6* → 112/81* 14/9* → 448/243* 21/20* → 56/45* 7/5* → 224/135*

6.6 Factor *6/5 (316 c)

*6/5 (316 c)

9/8* → 27/20* 3/2* → 9/5* 1/1* → 6/5* 4/3* → 8/5* 16/9* → 16/15* 32/27* → 64/45* 128/81* → 256/135*	45/32* → 27/16* 15/8* → 9/8* 5/4* → 3/2* 5/3* → 1/1* 10/9* → 4/3* 40/27* → 16/9* 160/81* → 32/27* 320/243* → 128/81* 45/28* → 27/14* 15/14* → 9/7* 10/7* → 12/7* 40/21* → 8/7* 80/63* → 32/21-* 320/189* → 64/63-*	7/4* → 21/20* 7/6* → 7/5* 14/9* → 28/15* 28/27* → 56/45* 112/81* → 224/135*

6.7 Factor *5/4 (386 c)

*5/4 (386 c)

27/20* → 27/16* 9/5* → 9/8* 6/5* → 3/2* 8/5* → 1/1* 16/15* → 4/3* 64/45* → 16/9* 256/135* → 32/27* 9/8* → 45/32* 3/2* → 15/8* 1/1* → 5/4* 4/3* → 5/3* 16/9* → 10/9* 32/27* → 40/27* 128/81* → 160/81* 256/243* → 320/243* 9/7* → 45/28* 12/7* → 15/14* 8/7* → 10/7* 32/21* → 40/21* 64/63* → 80/63* 256/189* → 320/189*		21/20* → 21/16* 7/5* → 7/4* 28/15* → 7/6* 56/45* → 14/9* 224/135* → 28/27*

*81/64 (408 c)

4/3* → 27/16* 16/9* → 9/8* 32/27* → 3/2* 128/81* → 1/1* 256/243* → 4/3* 16/15* → 27/20* 64/45* → 9/5-* 256/135* → 6/5* 32/21* → 27/14* 64/63* → 9/7* 256/189* → 12/7*	10/9* → 45/32* 40/27* → 15/8* 160/81* → 5/4* 320/243* → 5/3*	28/27* → 21/16* 112/81* → 7/4* 448/243* → 7/6*

*32/25 (427 c)

	45/32* → 9/5* 15/8* → 6/5* 5/4* → 8/5* 5/3* → 16/15* 10/9* → 64/45* 40/27* → 256/135*	

*9/7 (435 c)

	5/4* → 45/28* 5/3* → 15/14* 10/9* → 10/7* 40/27* → 40/21* 160/81* → 80/63* 320/243* → 320/189*	21/16* → 27/16* 7/4* → 9/8 7/6* → 3/2 14/9* → 1/1 28/27* → 4/3 112/81* → 16/9 448/243* → 32/27* 21/20* → 27/20* 7/5* → 9/5* 28/15* → 6/5* 56/45* → 8/5* 224/135* → 16/15*

*21/16 (471c)

9/7* → 27/16* 12/7* → 9/8* 8/7* → 3/2* 32/21* → 1/1* 64/63* → 4/3* 256/189* → 16/9* 1/1* → 21/16* 4/3* → 7/4* 16/9* → 7/6* 32/27* → 14/9* 128/81* → 28/27* 256/243* → 112/81* 8/5* → 21/20* 16/15* → 7/5* 64/45* → 28/15* 256/135* → 56/45*	15/14* → 45/32* 10/7* → 15/8* 40/21* → 5/4* 80/63* → 5/3* 320/189* → 10/9*	

6.8 Factor *4/3 (498 c)

*4/3 (498 c)

27/16* → 9/8* 9/8* → 3/2* 3/2* → 1/1* 1/1* → 4/3* 4/3* → 16/9* 16/9* → 32/27* 32/27* → 128/81* 128/81* → 256/243* 27/20* → 9/5* 9/5* → 6/5* 6/5* → 8/5* 8/5* → 16/15* 16/15* → 64/45* 64/45* → 256/135* 27/14* → 9/7* 9/7* → 12/7* 12/7* → 8/7* 8/7* → 32/21* 32/21* → 64/63* 64/63* → 256/189*	45/32* → 15/8* 15/8* → 5/4* 5/4* → 5/3* 5/3* → 10/9* 10/9* → 40/27* 40/27* → 160/81* 160/81* → 320/243* 45/28* → 15/14* 15/14* → 10/7* 10/7* → 40/21* 40/21* → 80/63* 80/63* → 320/189*	21/16* → 7/4* 7/4* → 7/6* 7/6* → 14/9* 14/9* → 28/27* 28/27* → 112/81* 112/81* → 448/243* 21/20* → 7/5* 7/5* → 28/15* 28/15* → 56/45* 56/45* → 224/135*

*27/20 (520 c)

27/16* → 27/20* 9/8* → 9/5* 16/9* → 6/5* 32/27* → 8/5* 128/81* → 16/15* 256/243* → 64/45*	15/8* → 45/32* 5/4* → 15/8* 5/3* → 9/8* 10/9* → 3/2* 40/27* → 1/1* 160/81* → 4/3* 320/243* → 16/9* 10/7* → 27/14* 40/21* → 9/7* 80/63* → 12/7* 320/189* → 8/7*	14/9* → 21/20* 28/27* → 7/5* 112/81* → 28/15* 448/243* → 56/45*

6.9 Factor *7/5 (583 c)

*7/5 (583 c)

27/14* → 27/20* 9/7* → 9/5* 12/7* → 6/5* 8/7* → 8/5* 32/21* → 16/15* 64/63* → 64/45* 256/189* → 256/135* 3/2* → 21/20* 1/1* → 7/5* 4/3* → 28/15* 16/9* → 56/45* 32/27* → 224/135*	45/28* → 9/8* 15/14* → 3/2* 10/7* → 1/1* 40/21* → 4/3* 80/63* → 16/9* 320/189* → 32/27* 15/8* → 21/16* 5/4* → 7/4* 5/3* → 7/6* 10/9* → 14/9* 40/27* → 28/27* 160/81* → 112/81* 320/243* → 448/243*	

*45/32 (590 c)

6/5* → 27/16* 8/5* → 9/8* 16/15* → 3/2* 64/45* → 1/1* 256/135* → 4/3* 1/1* → 45/32* 4/3* → 15/8* 16/9* → 5/4* 32/27* → 5/3* 128/81* → 10/9* 256/243* → 40/27* 8/7* → 45/28* 32/21* → 15/14* 64/63* → 10/7* 256/189* → 40/21*		28/15* → 21/16* 56/45* → 7/4* 224/135* → 7/6*

*64/45 (610 c)

27/16* → 6/5* 9/8* → 8/5* 3/2* → 16/15* 1/1* → 64/45* 4/3* → 256/135*	45/32* → 1/1* 15/8* → 4/3* 5/4* → 16/9* 5/3* → 32/27* 10/9* → 128/81* 40/27* → 256/243* 45/28* → 8/7* 15/14* → 32/21* 10/7* → 64/63* 40/21* → 256/189*	21/16* → 28/15* 7/4* → 56/45* 7/6* → 224/135*

6.10 Factor *10/7 (617 c)

*10/7 (617 c)

27/20* → 27/14* 9/5* → 9/7* 6/5* → 12/7* 8/5* → 8/7* 16/15* → 32/21* 64/45* → 64/63* 256/135* → 256/189* 9/8* → 45/28* 3/2* → 15/14* 1/1* → 10/7* 4/3* → 40/21* 16/9* → 80/63* 32/27* → 320/189*		21/20* → 3/2* 7/5* → 1/1* 28/15* → 4/3* 56/45* → 16/9* 224/135* → 32/27* 21/16* → 15/8* 7/4* → 5/4* 7/6* → 5/3* 14/9* → 10/9* 28/27* → 40/27* 112/81* → 160/81* 448/243* → 320/243*

*40/27 (680 c)

27/20* → 27/16* 9/5* → 9/8* 6/5* → 16/9* 8/5* →32/27* 16/15* →128/81* 64/45* →256/243* 27/14* →10/7* 9/7* →40/21* 12/7*→ 80/63* 8/7* → 320/189* 45/32* → 15/8* 15/8* →5/4* 9/8* → 5/3* 3/2* → 10/9* 1/1* → 40/27* 4/3* → 160/81* 16/9* → 320/243*		21/20* → 14/9* 7/5* → 28/27* 28/15* → 112/81* 56/45* → 448/243*

6.11 Factor *3/2 (702 c)

*3/2 (702 c)

9/8* → 27/16* 3/2* → 9/8* 1/1* → 3/2* 4/3* → 1/1* 16/9* → 4/3* 32/27* → 16/9* 128/81* → 32/27* 256/243* → 128/81* 9/5* → 27/20* 6/5* → 9/5* 8/5* → 6/5* 16/15* → 8/5* 64/45* → 16/15* 256/135* → 64/45* 9/7* → 27/14* 12/7* → 9/7* 8/7* → 12/7* 32/21* → 8/7* 64/63* → 32/21* 256/189* → 64/63*	15/8* → 45/32* 5/4* → 15/8* 5/3* → 5/4* 10/9* → 5/3* 40/27* → 10/9* 160/81* → 40/27* 320/243* → 160/81* 45/32* → 45/28* 10/7* → 15/14* 40/21* → 10/7* 80/63* → 40/21* 320/189* → 80/63*	7/4* → 21/16* 7/6* → 7/4* 14/9* → 7/6* 28/27* → 14/9* 112/81* → 28/27* 448/243* → 112/81* 7/5* → 21/20* 28/15* → 7/5* 56/45* → 28/15* 224/135* → 56/45*

*32/21 (729 c)

27/16* → 9/7* 9/8* → 12/7* 3/2* → 8/7* 1/1* → 32/21* 4/3* → 64/63* 16/9* → 256/189*	45/32* → 15/14* 15/8* → 10/7* 5/4* → 40/21* 5/3* → 80/63* 10/9* → 320/189*	21/16* → 1/1* 7/4* → 4/3* 7/6* → 16/9* 14/9* → 32/27* 28/27* → 128/81* 112/81* → 256/243* 21/20* → 8/5* 7/5* → 16/15* 28/15* → 64/45* 56/45* → 256/135*

*14/9 (765 c)

27/14* → 3/2* 9/7* → 1/1* 12/7* → 4/3* 8/7* → 16/9* 32/21* → 32/27* 64/63* → 128/81* 256/189* → 256/243* 27/16* → 21/16* 9/8* → 7/4* 3/2* → 7/6* 1/1* → 14/9* 4/3* → 28/27* 16/9* → 112/81* 32/27* → 448/243* 27/20* → 21/20* 9/5* → 7/5* 6/5* → 28/15* 8/5* → 56/45* 16/15* → 224/135*	45/35* → 5/4* 15/14* → 5/3* 10/7* → 10/9* 40/21* → 40/27* 80/63* → 160/81* 320/189* → 320/243*	

*25/16 (773 c)

9/5* → 45/32* 6/5* → 15/8* 8/5* → 5/4* 16/15* → 5/3* 64/45* → 10/9* 256/135* → 40/27*		

*128/81 (792 c)

27/16* → 4/3* 9/8* → 16/9* 3/2* → 32/27* 1/1* → 128/81* 4/3* → 256/243* 27/20* → 16/15* 9/5* → 64/45* 6/5* → 256/135* 27/14* → 32/21* 9/7* → 64/63* 12/7* → 256/189*	45/32* → 10/9* 15/8* → 40/27* 5/4* → 160/81* 5/3* → 320/243*	21/16* → 28/27* 7/4* → 112/81* 7/6* → 448/243*

6.12 Factor *8/5 (814 c)

*8/5 (814 c)

27/16* → 27/20* 9/8* → 9/5* 3/2* → 6/5* 1/1* → 8/5* 4/3-* → 16/15* 16/9* → 64/45* 32/27* → 256/135*	45/32* → 9/8* 15/8* → 3/2* 5/4* → 1/1* 5/3* → 4/3* 10/9* → 16/9* 40/27* → 32/27* 160/81* → 128/81* 320/243* → 256/243* 45/28* → 9/7* 15/14* → 12/7* 10/7* → 8/7* 40/21* → 32/21* 80/63* → 64/63* 320/189* → 256/189*	21/16* → 21/20* 7/4* → 7/5* 7/6* → 28/15* 14/9* → 56/45* 28/27* → 224/135*

6.13 Factor *5/3 (884 c)

*5/3 (884 c)

27/20* → 9/8* 9/5* → 3/2* 6/5* → 1/1* 8/5* → 4/3* 16/15* → 16/9* 64/45* → 32/27* 256/135* → 128/81* 27/16* → 45/32* 9/8* → 15/8* 3/2* → 5/4* 1/1* → 5/3* 4/3* → 10/9* 16/9* → 40/27* 32/27* → 160/81* 128/81* → 320/243* 27/14* → 45/28* 9/7* → 15/14* 12/7* → 10/7* 8/7* → 40/21* 32/21* → 80/63* 64/63* → 320/189*		21/20* → 7/4* 7/5* → 7/6* 28/15* → 14/9* 56/45* → 28/27* 224/135* → 112/81*

*27/16 (906 c)

16/9* → 3/2* 32/27* → 1/1 128/81* → 4/3 256/243* → 16/9* 8/5* → 27/20* 16/15* → 9/5* 64/45* → 6/5* 256/135* → 8/5* 32/21* → 9/7* 64/63* → 12/7* 256/189* → 8/7*	5/4* → 45/32* 10/9* → 15/8* 40/27* → 5/4* 160/81* → 5/3* 320/243* → 10/9* 40/21* → 45/28* 80/63* → 15/14* 320/189* → 10/7*	14/9* → 21/16* 28/27* → 7/4* 112/81* → 7/6* 448/243* → 14/9* 56/45* → 21/20* 224/135* → 7/5*

*128/75 (925 c)

	45/32* → 6/5* 15/8* → 8/5* 5/4* → 16/15* 5/3* → 64/45* 10/9* → 256/135*	

6.14 Factor *12/7 (933 c)

*12/7 (933 c)

9/8* → 27/14*	15/8* → 45/28*	21/16* → 9/8*
3/2* → 9/7*	5/4* → 15/14*	7/4* → 3/2*
1/1* → 12/7*	5/3* → 10/7*	7/6* → 1/1*
4/3* → 8/7*	10/9* → 40/21*	14/9* → 4/3*
16/9* → 32/21*	40/27* → 80/63*	28/27* → 16/9*
32/27* → 64/63*	160/81* → 320/189*	112/81* → 32/27*
128/81* → 256/189*		448/243* → 128/81*
		21/20* → 9/5*
		7/5* → 6/5*
		28/15* → 8/5*
		56/45* → 16/15*
		224/135* → 64/45*

6.15 Factor *7/4 (969 c)

*7/4 (969 c)

27/14* → 27/16* 9/7* → 9/8* 12/7* → 3/2* 8/7* → 1/1* 32/21* → 4/3* 64/63* → 16/9* 256/189* → 32/27* 3/2* → 21/16* 1/1* → 7/4* 4/3* → 7/6* 16/9* → 14/9* 32/27* → 28/27* 128/81* → 112/81* 256/243* → 448/243* 9/5* → 21/20* 8/5* → 7/5* 16/15* → 28/15* 64/45* → 56/45* 256/135* → 224/135*	15/14* → 15/8* 10/7* → 5/4* 40/21* → 5/3* 80/63* → 10/9* 320/189* → 40/27*	27/14* → 27/16* 9/7* → 9/8* 12/7* → 3/2* 8/7* → 1/1* 32/21* → 4/3* 64/63* → 16/9* 256/189* → 32/27* 3/2* → 21/16* 1/1* → 7/4* 4/3* → 7/6* 16/9* → 14/9* 32/27* → 28/27* 128/81* → 112/81* 256/243* → 448/243* 9/5* → 21/20* 8/5* → 7/5* 16/15* → 28/15* 64/45* → 56/45* 256/135* → 224/135*

6.16 Factor *16/9 (996 c)

*16/9 (996 c)

27/16* → 3/2* 9/8* → 1/1* 3/2* → 4/3* 1/1* → 16/9* 4/3* → 32/27* 16/9* → 128/81* 32/27* → 256/243* 21/20* → 6/5* 9/5* → 8/5* 6/5* → 16/15* 8/5* → 64/45* 16/15* → 256/135* 27/14* → 12/7* 9/7* → 8/7* 12/7* → 32/21* 8/7* → 64/63* 32/21* → 256/189*	45/32* → 5/4* 15/8* → 5/3* 5/4* → 10/9* 5/3* → 40/27* 10/9* → 160/81* 40/27* → 320/243* 45/28* → 40/21* 10/7* → 80/63* 40/21* → 320/189*	21/16* → 7/6* 7/4* → 14/9* 7/6* → 28/27* 14/9* → 112/81* 28/27* → 448/243* 21/20* → 28/15* 7/5* → 56/45* 28/15* → 224/135*

6.17 Factor *9/5 (1018 c)

*9/5 (1018 c)

3/2* → 21/20* 1/1* → 9/5* 4/3* → 6/5* 16/9* → 8/5* 32/27* → 16/15* 128/81* → 64/45* 256/243* → 256/135*	15/8* → 27/16* 5/4* → 9/8* 5/3* → 3/2* 10/9* → 1/1* 40/27* → 4/3* 160/81* → 16/9* 320/243* → 32/27* 15/14* → 27/14* 10/7* → 9/7* 40/21* → 12/7* 80/63* → 8/7* 320/189* → 32/21*	7/6* → 21/20* 14/9* → 7/5* 28/27* → 28/15* 112/81* → 56/45* 448/243* → 224/135*

*28/15 (1081 c)

27/14* → 9/5* 9/7* → 6/5* 12/7* → 8/5* 8/7* → 16/15* 32/21* → 64/45* 64/63* → 256/135* 9/8* → 21/20* 3/2* → 7/5* 1/1* → 28/15* 4/3* → 56/45* 16/9* → 224/135*	45/28* → 3/2* 15/14* → 1/1* 10/7* → 4/3* 40/21* → 16/9* 80/63* → 32/27* 320/189* → 128/81* 45/32* → 21/16* 15/8* → 7/4* 5/4* → 7/6* 5/3* → 14/9* 10/9* → 28/27* 40/27* → 112/81* 160/81* → 448/243*	

6.18 Factor *15/8 (1088 c)

*15/8 (1088 c)

9/5* → 27/16* 6/5* → 9/8* 8/5* → 3/2* 16/15* → 1/1* 64/45* → 4/3* 256/135* → 16/9* 3/2* → 45/32* 1/1* → 15/8* 4/3* → 5/4* 16/9* → 5/3* 32/27* → 10/9* 128/81* → 40/27* 256/243* → 160/81* 12/7* → 45/28* 8/7* → 15/14* 32/21* → 10/7* 64/63* → 40/21* 256/189* → 80/63*		7/5* → 21/16* 28/15* → 7/4* 56/45* → 7/6* 224/135* → 14/9*

7. Chord lists

In this chapter, we suggest a list of chords to be used in the context of a modulation[1]. Chords are organized according to the following rules to produce a meaningful and constant classification:

— by ascending order of limit (the highest prime number in the chord);
— by ascending order of the second highest prime number (for example, 3-5-7 is listed after 3-7-9, because its second highest prime number is 5, compared to 3 for 3-7-9);
— by ascending order of the total of all identities of the chord (for example, the chord 3-5-7 is listed after the chord 5-7, since the total of 3 + 5 + 7 is greater than that of 5 + 7).

We eliminated some chord arrangements that sound too dissonant and included others with reservations. These subjective rulings were guided by one major, objective criterium already stated in the chapter on modulation: the more the harmonics of a chord are organized in ascending order of identity, the greater the consonance of that chord. Furthermore, disorganized harmonics make it harder to perceive their identifications. For example, reversing 1- and 9- identities produces a [9 : 16] interval, which tends to be perceived as the similar [4 : 7] interval whose identites are well organized.

In the example below for the chord 3-15, it would be best to replace disorganized arrangements (in grey) with better organized ones (in colour).

	16	15	24	32	15	24
3-15	15	8	16	24	12	15
	12	6	15	15	8	8

Table 7.1: Various arrangements of the 3-15 chord

[1] Since chords that simply permute harmonics within a single tonality allow for more complexity, listing them all seemed useless to us, impossible even.

In this other example for the chord 5-9, arrangements [9 : 16 : 20] and [5 : 9 : 16] have been taken out, which also shows how difficult they were to integrate into satisfying modulations in the *Supplements*.

3-15	10 9 8	9 5 4	16 10 9		9 8 5	

Table 7.2: Various arrangements of the 5-9 chord

7.1 3-limit chords

3	4 3 2	3 2 1	4 3 2	3 2 1	4 3 2	3 2 1
9	16 9 8	9 8 4	16 9 8	9 8 4	16 9 8	9 8 4
3-9	12 9 8	9 6 4	16 12 9		9 8 6	16 9 6

7.2 5-limit chords

5	8 5 4	5 4 2	8 5 4	5 4 2	8 5 4	5 4 2
3-5	6 5 4	5 3 2	8 6 5	12 8 5	5 4 3	8 5 3
5-9	10 9 8	9 5 4	16 10 9		9 8 5	
3-5-9	12 10 9	20 12 9	16 9 5	9 8 5	10 9 6	9 5 3
15	16 15 8	15 8 4	16 15 8	15 8 4	16 15 8	15 8 4
3-15	16 15 12	15 8 6	24 16 15	32 24 15	15 12 8	24 15 8
5-15	20 16 15	32 20 15	15 10 8	20 15 8	16 15 10	15 8 5
3-5-15	15 12 10	24 15 10	20 15 12	15 10 6	24 20 15	40 24 15
3-9-15	18 15 12	15 9 6	24 18 15	36 24 15	15 12 9	24 15 9
5-25	25 20 16	40 25 16	32 25 20	25 16 10	40 32 25	64 40 25

7.3 7-limit chords

7	8 7 4	7 4 2	8 7 4	7 4 2	8 7 4	7 4 2
3-7	8 7 6	7 4 3	12 8 7	16 12 7	7 6 4	12 7 4
7-9	9 8 7	18 16 7	14 9 8	9 7 4	16 14 9	
3-7-9	9 7 6	14 9 6	12 9 7	18 12 7	14 12 9	24 14 9
5-7	10 8 7	16 10 7	7 5 4	10 7 4	8 7 5	14 8 5
3-5-7	7 6 5	12 7 5	10 7 6	7 5 3	12 10 7	20 12 7
5-7-9	10 9 7		14 10 9		9 7 5	14 9 5
7-15	16 15 14				15 14 8	
3-7-15	15 14 12			15 12 7		
5-7-15		15 10 7			15 14 10	

7.4 11-limit chords

11	16 11 8	11 8 4	16 11 8	11 8 4	16 11 8	11 8 4
3-11	12 11 8	11 6 4	16 12 11		11 8 6	16 11 6
9-11	11 9 8					
5-11	11 10 8		16 11 10	11 8 5		
3-5-11	12 11 10	11 6 5			11 10 6	
5-9-11	11 10 9			11 9 5		
7-11	11 8 7		14 11 8	11 7 4		
3-7-11	14 12 11		11 7 6	14 11 6		

7.5 13-limit chords

13	16 13 8	13 8 4	16 13 8	13 8 4	16 13 8	13 8 4
3-13	16 13 12	13 8 6	24 16 13		13 12 8	
9-13			13 9 8			
3-9-13				13 9 6		
5-13			13 10 8	20 13 8	16 13 10	13 8 5
3-5-13	13 12 10		20 13 12	13 10 6		
5-9-13				13 9 5		
7-13	16 14 13				14 13 8	13 7 4
3-7-13	14 13 12				13 12 7	

7.6 17-limit chords

17	32 17 16	17 16 8	32 17 16	17 16 8	32 17 16	17 16 8
3-17	24 17 16	17 12 8	32 24 17		17 16 12	32 17 12
9-17	18 17 16					
3-9-17	24 18 17				18 17 12	
5-17	20 17 16	17 10 8			17 16 10	
3-5-17	24 20 17		17 12 10	24 17 10	20 17 12	17 10 6
7-17	17 16 14		28 17 16	17 14 8		
3-7-17	17 14 12	28 17 12	24 17 14	17 12 7	28 24 17	
5-7-17	20 17 14	17 10 7	28 20 17	40 28 17	17 14 10	28 17 10

7.7 19-limit chords

19	32 19 16	19 16 8	32 19 16	19 16 8	32 19 16	19 16 8
3-19	24 19 16	19 12 8	32 24 19	48 32 19	19 16 12	32 19 12
9-19	19 18 16	36 19 16				
3-9-19	24 19 18				19 18 12	
5-19	20 19 16	19 10 8			19 16 10	
3-5-19	24 20 19		19 12 10	24 19 10	20 19 12	19 10 6
15-19	19 16 15		30 19 16	19 15 8	32 15 19	60 32 19
3-15-19	30 24 19	48 30 19	19 15 12	30 19 12	24 19 15	38 24 15

7.8 23-limit chords

23	32 23 16	23 16 8	32 23 16	23 16 8	32 23 16	23 16 8
3-23	24 23 16	23 12 8			23 16 12	32 23 12
5-23	23 20 16	40 23 16		23 16 10		
3-5-23	24 23 20	23 12 10			23 20 12	

Conclusion

Organizational structures hide the phenomena on which they are built. For example, spoken language requires both a certain perception of fundamental elements (phonemes), and a state of transcendence that enables us to forget these very elements. Similarly, in the aural sphere, music is the negation of sound for the benefit of its organization. Nevertheless, a sound that is completely left to its autonomous state (as opposed to it being collected for its potential as a poetic tool) goes back to the hum of nature. Thus, in order for it to be aesthetically perceivable, sound needs music as much as music needs sound.

This dialectic tension is found at the heart of numbers-based harmony, whose material, proportions of integers, corresponds to the exact measurement of the phenomenon. A balance becomes possible between music and sound, between order—necessary for complexity to emerge—and the infinite creative potential of chaos.

May we not forget it again.

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Author’s bio

Simon Martin (b.1981 in Rouyn-Noranda, Quebec) is a composer of contemporary concert music. He lives in Montreal.

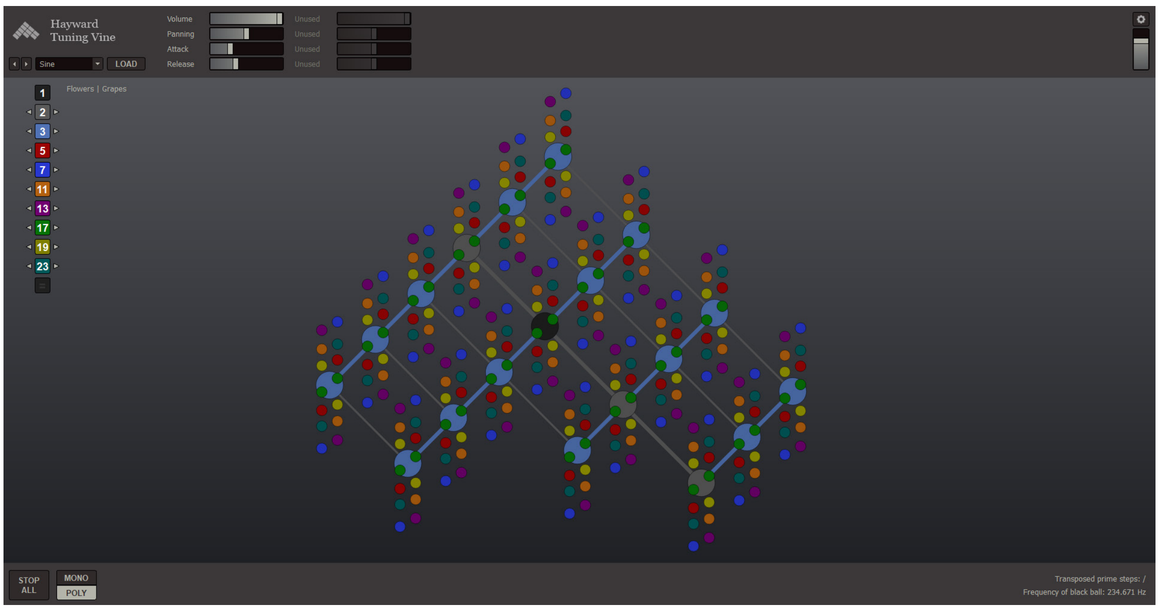
His concert-length works have been performed, through his own creative organization Projections libérantes founded in 2011, by the world-renowned ensembles Quatuor Bozzini (2015), Zinc & Copper (2017), and Ensemble Musikfabrik (2019, 2021).

In 2016, Simon Martin won the Opus Award for Best Composer with a $10,000 grant from the Conseil des arts et des lettres du Québec. He is currently completing a three-year (2019-2022) postdoctoral internship in research and creation at Concordia University’s matralab centre with support from Fonds de recherche du Québec.

Simon Martin’s music, imbued with interiority, makes us witnesses to a mystery that questions us. The narrative of his work/concert is chiselled directly from the subject matter, leading to a quest for the perfection of harmony which simultaneously becomes a wellspring of beauty and drama.

Appendix: The Hayward Tuning Vine Software

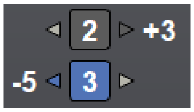
We recommend using the Hayward Tuning Vine software with this *Treatise* and its *Supplements*. This is the software's only window as seen upon launching the application:



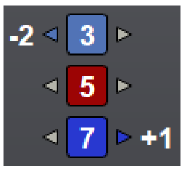
To reproduce any modulation, you need to open two windows, one for each tonality. First, to produce the starting tonality in the first window, you must consider its factors. For example, in the case of the 256/243* tonality, the factors are $1/3^5$. Then, you need to click on the right arrow (larger) of the prime number of the numerators and the left arrow (smaller) of the prime numbers of the denominator. For $1/3^5$, you would not need any right arrows (since the numerator is 1), but you would need to click five times on the left arrow of number 3 (since the denominator of the tonality is 3^5).

[1] A free trial version is available at tuningvine.com (last viewed 2 March 2021). An updated version is also available. It gives access to the 29- and 31-harmonics, among other new features.

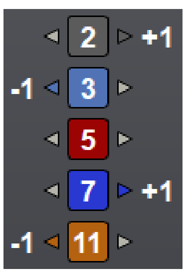
Finally, you would have to use the number 2 to program the octave desired.



Another example: to produce the 14/9* tonality (factors 7/3^2), you would click once on the right arrow of the number 7 and twice on the left arrow of number 3.

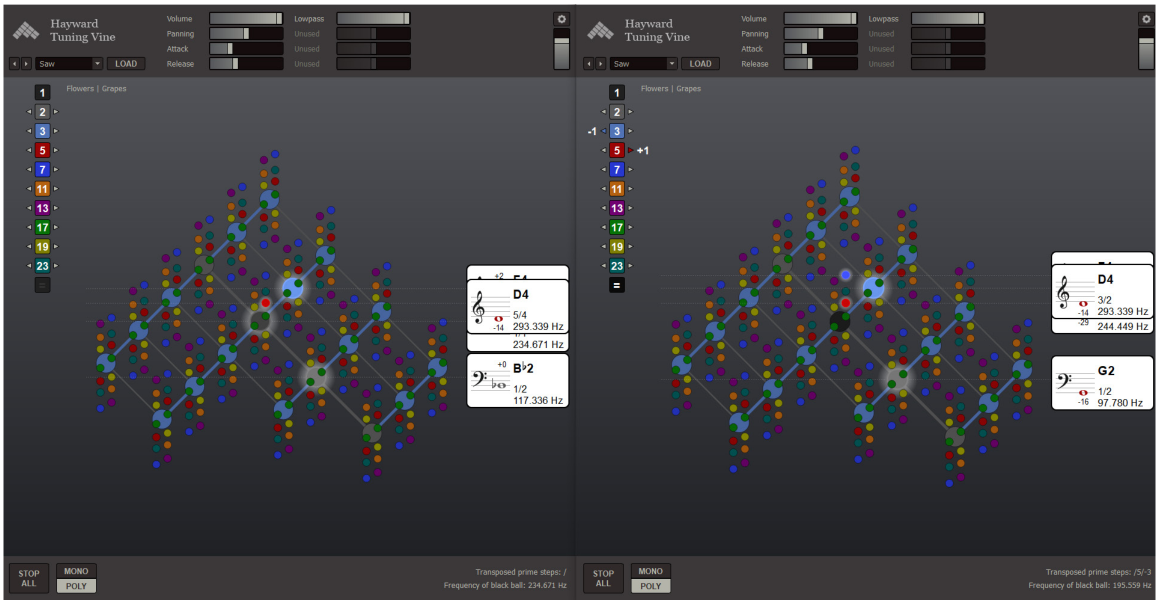


Starting with the 14/9* tonality, you can modulate in the same window, let us say by a factor of *12/11 (factors 3/11) by clicking on the right arrow of number 3 and the left arrow of number 11 to produce the 56/33* tonality (factors 7/3*11).



One final example: this is the table for a sequence between chords 3-5 → 3-5-7 by a *5/3 modulation factor, and how to reproduce it in the application using two windows programmed to tonalities 1/1* and 5/3* respectively.

	*5/3 (884 c)
6 ↘ 7	36 : 35 (-49)
5 = 6	
4 ↗ 5	24 : 25 (+71)
2 ↘ 2	6 : 5 (-316)


Hayward Tuning Vine
Volume
Panning
Attack
Release
Lowpass
Unused
Saw
LOAD
Flowers | Grapes
D4
5/4
293.339 Hz
234.671 Hz
B♭2
1/2
117.336 Hz
STOP ALL
MONO
POLY
Transposed prime steps: /
Frequency of black ball: 234.671 Hz
D4
3/2
293.339 Hz
244.449 Hz
G2
1/2
97.780 Hz
Transposed prime steps: /5/-3
Frequency of black ball: 195.559 Hz

$$\frac{1}{T}$$